Generalization of Waveguide Invariants and Application to Passive Time Reversal

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Abstract

In most underwater acoustic experiments acoustic sources and hydrophone arrays are moored so as to provide a geometry as controllable as possible. A more operational approach is to use moving sources and drifting acoustic receivers in which case the data exhibits continuous phase and amplitude changes due to depth and range shifts. This may be problematic when the processing of the collected acoustic data requires the use of correlation between successive received signals, e.g., in passive time reversal where a probe-signal is sent ahead of the data for post crosscorrelation. This paper demonstrates that in the passive time reversal context the source-array range, the array and source depth mismatches that occurs during the data transmission can be compensated using an appropriate frequency shift of the received probe-signal pressure field. Acoustic simulations and real data collected during the MREA’04 experiment shows that the frequency translation required for the geometric mismatch compensation can be computed using invariant properties of the waveguide, and thus provide a potential for substantial processing signal to noise ratio gain in underwater communications between moving platforms.

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I. Introduction

Active Time reversal (aTR) has been shown to produce temporally and spatially focused acoustic signals in a stationary environment. Such focusing capabilities are a consequence of the time reversal invariance of the linear lossless wave equation describing acoustic propagation in the ocean environment [1] and one of its major goals is the development of underwater coherent communication techniques since focusing is similar to undoing the multipath structure of the channel [2, 3]. Passive Time Reversal (pTR), originally referred to as Passive Phase Conjugation (PPC) [4], is a synthetic version of aTR where a probe-signal is transmitted ahead of the data-signal in order to provide an estimate of the underwater channel Impulse Responses (IRs). Time focusing is then performed at the array side by simply convolving a time reversed version of the estimated IRs with the incoming data-signal [5, 6, 7].

In the presence of a moving source and a free drifting array, there exist three major sources of mismatch: the source-array range shift, the source depth shift and the array depth shift. Due to those geometric mismatches pTR rapidly loses its time focusing capability [8, 9] and up to now there has been no attempt to incorporate a geometric tracking in pTR in order to generate a long-term focusing.

For aTR, previous work by Song el al. [10] addresses the focal spot range shift problem using a frequency translation of the array received acoustic field. The technique can be readily applied in pTR, making it possible to perform source-array range tracking. It is based on the $\beta$ waveguide invariant [11] and only accounts for shifting in range of the focal spot intensity. In what concerns the focal spot depth shift in aTR a different strategy was proposed by Walker [12]. In Walker’s work a depth shift of the focal spot has been achieved, but contrarily to the source-array range shift proposed by Song it is not based in simple waveguide invariant properties, and its implementation for the pTR source depth shift compensation in real time does not look straightforward.
In [13] the authors presented preliminary simulations and real data results suggesting that similarly to the source-array range shift compensation, the source depth shift and the array depth shift compensation could be performed by a frequency shift of the acoustic field. It was found that for narrowband signals with a center frequency of 3.6 kHz, frequency shift compensation performs well in the vicinity of the geometric canonical values. Moreover, it was found that associated with the intensity geometric mismatch compensation there is an approximately linear phase. In the present paper the theoretical proof for such compensation capability is given for the source-array range shift, for the array depth and source depth shifts.

The source-array range compensation depends on the waveguide invariant $\beta$ that relates the modal horizontal phase velocity with the horizontal group velocity. The invariant $\beta$ summarizes in a single parameter the dispersive characteristics of the acoustic field in a waveguide. In fact, it has been shown [11] that the lines of constant sound intensity, constant rate of change of the phase velocity along the waveguide, and constant envelope group delay have constant slope $\beta$ in the frequency/range plane. In this paper by using a perfect waveguide a different interpretation of the invariant is given in order to explicitly derive an approximation of the horizontal wavenumber by using the group slowness. Such reinterpretation of $\beta$ is then used for range shift compensation of the pressure field in intensity and phase by simply using an appropriate frequency shift. Using a similar approach it was found that the vertical wave number can also be approximated by considering a frequency invariant $\zeta$ that relates the vertical phase velocity with the horizontal group velocity, which allows for the compensation of source and array depth shifts in pTR applications.

When applied to underwater communications the proposed geometric mismatch compensation method provides a longer stability of a pTR communications system by increasing the elapsed time between probe-signal transmissions, but also makes it possible to estimate source and receiving array depths oscillation, and source-receiving range variations, under the form of an environmental equalizer [14].
Section II. explains how the invariant $\beta$ can be used to compute the horizontal wave number and using a similar strategy derives a new invariant $\zeta$ that can be used to compute the vertical wavenumber using the horizontal group slowness. Section III. explains the influence of the geometric mismatch over the pTR in a perfect waveguide and finds compensation strategies using a frequency shift that can be computed with the invariants $\beta$ and $\zeta$. Section V. heuristically extends the findings to realistic environments, by taking into consideration the WKB approximation and the Pekeris waveguide. Section VI. uses real data, narrowband signals centered at 3.6 kHz, with geometric mismatch to demonstrate the usefulness of the new findings, and shows that longer stability of the pTR processor can be attained at least up to a range mismatch of 25 m, and a source depth mismatch and an array depth mismatch of about 0.7 m. Section VII. summarizes the compensation procedure and describes possible applications.

II. The waveguide invariants

In the this section it will be shown that the horizontal wavenumber $k_m$ and the vertical wavenumber $\gamma_m$ can be computed by a linear approximation of the horizontal group slowness and that such approximation is made possible by the frequency invariants $\beta$ and $\zeta$, respectively. These results will be obtained for the perfect waveguide and extended heuristically to the Pekeris waveguide and to the WKB approximation in Section V.

The derivation draws upon generic results proposed in Appendix A, where it is shown that one monotonic function $\Phi$ can be linearly approximated by another monotonic function $\Pi$ using a least-squares approximation or by setting one point of the two functions to the same position and then rotating one of the functions until it fits the other at a different point. The later provides a connection with the current waveguide invariant theory and the former becomes more useful in the context of the pTR geometric mismatch compensation proposed in this paper. The former strategy is a particular case of the later and ensures a
smaller approximation error.

A. Approximation of the horizontal wavenumbers using waveguide invariants

The two strategies of Appendix A can be applied to the approximation of horizontal wavenumbers by their reciprocals by setting $\Phi_m = k_m$ and $\Pi_m = k_m^{-1}$. Using (A4) it results that $k_m$ can be approximated by $k'_m$ with

$$k'_m = -\beta'_{\mu,\nu} k_m^{-1} + \rho'_{\beta,\mu,\nu},$$  \hspace{1cm} (1)

where $m = \mu$ and $m = \nu$ are the modes where $k_m$ and $k'_m$ coincide,

$$\beta'_{\mu,\nu} = \frac{k_\mu - k_\nu}{\frac{1}{k_\mu} - \frac{1}{k_\nu}},$$  \hspace{1cm} (2)

and

$$\rho'_{\beta,\mu,\nu} = k_\nu + \frac{1}{\beta'_{\mu,\nu} k_\nu}.$$  \hspace{1cm} (3)

In a perfect waveguide the horizontal wavenumber is given by

$$k_m = \sqrt{\frac{\omega^2}{c^2} - \gamma_m^2},$$  \hspace{1cm} (4)

where $\gamma_m$ is the vertical wavenumber (which, in a range independent waveguide, is frequency independent). Using (4) the horizontal group slowness becomes

$$\frac{1}{u_{h,m}} = \frac{d k_m}{d \omega} = \frac{\omega}{c^2} \frac{1}{k_m},$$  \hspace{1cm} (5)

and the horizontal phase slowness

$$\frac{1}{v_{h,m}} = \frac{k_m}{\omega}.$$  \hspace{1cm} (6)

Multiplying and dividing the fist term of the right hand side of (1) by $(\omega/c)^2$, multiplying and dividing the second term by $\omega$, and considering the phase and the group slowness equations for the perfect waveguide (5) and (6), respectively, it results

$$k'_m = -\beta_{\mu,\nu} \omega \frac{d k_m}{d \omega} + \omega \rho_{\beta,\mu,\nu},$$  \hspace{1cm} (7)
where
\[ \beta_{\mu,\nu} = -\frac{k_{\mu}}{c^2 k_{\mu}} - \frac{k_{\nu}}{c^2 k_{\nu}} = -\frac{1}{v_{h,\mu}} - \frac{1}{u_{h,\nu}}, \] (8)
and
\[ \rho_{\beta,\mu,\nu} = \frac{1}{v_{h,\nu}} + \beta_{\mu,\nu} \frac{1}{u_{h,\mu}}. \] (9)
Since (7) can be rewritten as
\[ \frac{k_m}{\omega} \approx -\beta_{\mu,\nu} \frac{d k_m}{d \omega} + \omega \rho_{\beta,\mu,\nu}, \] (10)
it corresponds to linearly approximate the horizontal phase slowness using the horizontal group slowness [15] \(^1\), and \(\beta_{\mu,\nu}\) is usually termed waveguide invariant [11] [16]. Here it will be re-termed as horizontal waveguide invariant since it relates the horizontal phase slowness with the horizontal group slowness.

In a similar manner the approximation can be done considering the least-squares form (A1), that can be applied by considering all modes or just a subset \(M_e\) of the propagating modes \(M\), resulting in
\[ k_m' = -\beta_{e,\omega} \frac{d k_m}{d \omega} + \omega \rho_{\beta,e}, \] (11)
where
\[ \beta_e = \frac{k_m k_m^{-1} - k_m^{-1} k_m^{-1} c^2}{(k_m^{-1})^2 - k_m^{-1} c^2 \omega^2}; \] (12)
and
\[ \rho_{\beta,e} = \frac{1}{v_{h,m}} + \beta_e \frac{1}{u_{h,m}}. \] (13)
where the bar represents the mean over the assumed subset of modes \(M_e\).

Since (7) represents a set of linear approximations to \(k_m\) it is expected that the approximation in the least-squares error sense (11) its one or close to one of them. In other words, there should exist an effective number of modes \(M_e\) and a pair \((\mu, \nu)\) where \(\beta_e \approx \beta_{\mu,\nu}\) and

\(^1\)Figure 1 of reference [15] shows that there is an approximately linear relation between phase speed and group speed. A case with two groups of modes that result in a dual slope linear relation is presented.
\[ \rho_{\beta,e} \approx \rho_{\beta,\mu,\nu} \] The approximation using an effective number of modes \( M_e < M \) is plausible since in a real situation the waveguide itself filters the higher order modes or, at least, strongly attenuates them. Such filtering corresponds, in the ray mode analogy, to eliminating rays with steeper angles.

In order to develop signal processing techniques that make use of the horizontal wavenumber approximation \( k'_m \) it is important to demonstrate the frequency invariance of \( \beta_e \) and \( \rho_{\beta,e} \). To establish that property the following auxiliary normalized product will be used

\[
\Gamma_{\beta,m} = \frac{k_m k_m}{\gamma_m^2 + k_m^2}, \quad (14)
\]

where \( (\gamma_m^2 + k_m^2) = (\omega/c)^2 \) is the wavenumber absolute value.

In the ray mode analogy it is considered by Tolstoy ([17] pp. 102) that the ray solution to the wave equation defines an infinite number of angles corresponding to the angles of incidence. The mode solution defines a finite number of angles that correspond to the rays that reinforce each other. So, in a perfect waveguide each mode is associated to an angle from the horizontal \( \pm \theta_m \) that corresponds to an angle of incidence \( (\pi/2 - \theta_m) \). In such context the horizontal and the vertical wavenumbers can be defined as

\[
k_m = \frac{\omega}{c} \cos \theta_m, \quad \gamma_m = \frac{\omega}{c} \sin \theta_m, \quad (15)
\]

where \( c \) represents the waveguide sound speed (assumed isovelocity). By using (15), product (14) becomes

\[
\Gamma_{\beta,m} = \Gamma_{\beta}(\theta_m) = \cos^2(\theta_m) \quad (16)
\]

where \( \theta_m \in [0, \pi/2] \). Figure 1 shows the product \( \Gamma_{\beta,m} \) as a function of \( \theta_m \) (dotted curve). As the frequency increases the angles \( \theta_m \) shift to the left and new angles, that correspond to new modes/rays, are included every \( \omega_{0,m} = (c/D)m \pi \) ([17] pp. 99), where \( \omega_{0,m} \) is the mode \( m \) cutoff frequency. Although \( \theta_m \) changes with frequency, for a sufficiently high number of propagating modes \( M \) the shape of \( \Gamma_{\beta,m} \) remains unchanged, in particular for small values
of \( m \) where \( \theta_m \) is densely populated. That means that the shape of \( \Gamma_{\beta,m} \) becomes invariant with increasing frequency.

\[
\Gamma'_{\beta,m} = \frac{k'_m k_m}{\gamma_m^2 + k_m^2}, \\
\Gamma'(\theta_m) = -\beta_e + \rho_{\beta,e} \cos \theta_m, \tag{17}
\]

where \( k'_m \) is given by (11). This is shown by the circles in Figure 1 for \( M_e = M/2 \) corresponding to \( \theta_m \in ]0, \pi/6[ \). Since the \( \Gamma_{\beta,m} \) shape is invariant with frequency its approximation \( \Gamma'_\beta(\theta_m) \) will be almost invariant, and that makes \( \beta_e \) and \( \rho_{\beta,e} \) also almost frequency invariant. In fact as the frequency increases \( M_e \) increases and \( \beta_e \) and \( \rho_{\beta,e} \) oscillate around a frequency independent mean value with an amplitude that decreases with frequency.

**B. Approximation of the vertical wavenumbers using waveguide invariants**

The approximation of the vertical wavenumber using the horizontal wavenumber inverse is analogous to the approximation of the horizontal wavenumber of Section A. and is straight-
forward considering the two linear approximation strategies of Appendix A with $\Phi_m = \gamma_m$ and $\Pi_m = k_m^{-1}$.

Considering the vertical phase slowness

$$\frac{1}{v_{v,m}} = \frac{\gamma_m}{\omega},$$

(18)

and the horizontal group slowness for the perfect waveguide (5), it results that $\gamma_m$ can be approximated by $\gamma'_m$

$$\gamma'_m = -\zeta_{\mu,\nu} \frac{d}{d\omega} k_m^0 + \omega \rho_{\xi,\mu,\nu},$$

(19)

where

$$\zeta_{\mu,\nu} = -\frac{\gamma_m}{\omega} \frac{\gamma_v}{\omega} = -\frac{1}{u_{h,\mu}} - \frac{1}{u_{h,\nu}},$$

(20)

and

$$\rho_{\xi,\mu,\nu} = \frac{1}{v_{v,\nu}} + \zeta_{\mu,\nu} \frac{1}{u_{h,\mu}},$$

(21)

where (20) defines a constant that will be called vertical waveguide invariant and its invariance with the frequency will be shown at the end of this section.

Similarly to the $k_m$ approximation in the least-squares sense, the approximation can be done considering the least-squares form (A1), yielding

$$\gamma'_m = -\zeta_{\xi,\omega} \frac{d}{d\omega} k_m^0 + \omega \rho_{\xi,\xi},$$

(22)

where

$$\zeta_{\xi} = -\frac{\gamma_m}{\omega} \frac{k_m^{-1}}{k_m^{-1}} \frac{c^2}{\omega^2},$$

(23)

and

$$\rho_{\xi,\xi} = \frac{1}{v_{v,m}} + \zeta_{\xi} \frac{1}{u_{h,m}}.$$

(24)

Since (19) represents a set of linear approximations to $\gamma_m$ it is expected that the approximation in the least-squares sense (22) will be close to one of them, that is, there is an effective number of modes $M_\xi$ and one pair $(\mu, \nu)$ such that $\zeta_{\xi} \approx \zeta_{\mu,\nu}$ and $\rho_{\xi,\xi} \approx \rho_{\zeta,\mu,\nu}$. 
In order to develop signal processing techniques that make use of the horizontal wavenumber approximation $\gamma'_m$ with wideband signals it is important to demonstrate the frequency invariance of $\zeta_e$ and $\rho_{\zeta,e}$. Such frequency invariance is better understood by considering the normalized product

$$\Gamma_{\zeta,m} = \frac{\gamma_m k_m}{\gamma_m^2 + k_m^2}. \quad (25)$$

Considering the ray mode analogy ([17] pp. 102) and $k_m$ and $\gamma_m$ given by (15), product (25) becomes

$$\Gamma_{\zeta,m} = \Gamma_{\zeta}(\theta_m) = \frac{1}{2} \sin(2\theta_m), \quad (26)$$

where $\theta_m \in [0, \pi/2]$. It can be approximated by $\Gamma'_{\zeta,m}$, defined as

$$\Gamma'_{\zeta}(\theta_m) = -\zeta_e + \rho_{\zeta,e} \cos \theta_m. \quad (27)$$

Figure 2 shows the product $\Gamma_{\zeta,m}$ as a function of $\theta_m$ (dotted line). The maximum of $\Gamma_{\zeta}$ is always at $M/\sqrt{2}$ that corresponds to $\theta_{M/\sqrt{2}} \approx \pi/4$. In Figure 2 the circles represent the approximation to $\Gamma_{\zeta}$ with $M_e = M/2$ that correspond to $\theta_m \in [0, \pi/6]$.

![Figure 2: Normalized product $\Lambda_{\zeta,m}$ (26) (dotted line), and its least-squares approximation (27) for an effective number of modes $M_e = M/2$ (circles).](image)
It is clear from Figures 1 and 2 that the approximation $\gamma'_m$ is poorer than $k'_m$, nevertheless in both cases the approximation quality can be enhanced by applying the approximation to a small number of modes $M_e$ or by applying the approximation in small sets of modes, resulting in different values of the waveguide invariants for each set. Still similar arguments as those used in Figure 1 demonstrate that $\zeta_e$ and $\rho_{\zeta,e}$ are also nearly invariant with frequency.

III. Geometric mismatch compensation in passive time reversal

This section establishes how the geometric mismatch (source-array range, source depth and array depth mismatch) and the frequency shift affect the pTR output in a perfect waveguide. The usefulness of the approximations to $k_m$ and $\gamma_m$ developed in Section II. for the geometric mismatch compensation will then arise naturally.

A. Passive Time Reversal in a stationary geometry

The behavior of pTR in a stationary environment was originally defined in [18] and latter refined by several authors, and it will be repeated here only for the definition of terms and for formal further understanding of how geometric mismatch affects pTR performance.

The pressure field received by each hydrophone of a Vertical Line Array (VLA) from a monochromatic point source is given by the Green’s function

$$G_\omega(R, z_0, z_i) = -\frac{j}{\rho \sqrt{8\pi R}} e^{-\frac{j \pi}{4}} \sqrt{k_m} \sum_{m=1}^{M} Z_m(z_i)Z_m(z_0) e^{ik_m R},$$

(28)

where $R$ represents the range between the source and the VLA, $z_0$ the source depth, $z_i$ the array hydrophones depth, $\rho$ the water density considered to be an unitary constant all over the water column, $Z_m$ is the depth dependent mode shape and $k_m$ is the horizontal wavenumber.

When the pTR processor is implemented in a stationary environment a first signal (index $m$) is sent from the source to the array and then the received pressure field is correlated with
a second transmission (index \(n\)). The resulting pressure field in the frequency
domain is given by

\[
P_{pc}(R, z_0, z_i, \omega) = \sum_{i=1}^{I} G_{\omega}(R, z_0, z_i) G_{\omega}^*(R, z_0, z_i)
\]

\[
= \frac{1}{\rho^2 8\pi R} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{Z_m(z_0)Z_n(z_0)}{\sqrt{k_m k_n}} \Psi(m, n) e^{j(k_m R - k_n R)},
\]

(29)

where, according to the basic assumptions of aTR \(^2\), the mode orthogonality property can be applied, such that

\[
\Psi(m, n) = \sum_{i=1}^{I} Z_m(z_i)Z_n(z_i) \approx \delta_{m,n}.
\]

(30)

Due to (30) the two summations in (29) can be replaced by a single one, the exponential vanishes, and the pTR pressure field in the frequency domain becomes approximately constant

\[
P_{pc}(\cdot) = \frac{1}{\rho^2 8\pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|}.
\]

(31)

In the following the problem of performance loss of pTR with variable geometry will be addressed and compensation strategies will be proposed. Since pTR performs the crosscorrelation between the first and a second received signal, it will be considered that the geometric mismatch affects the second received signal (index \(n\)) while the compensation strategies will be implemented in the first received signal (index \(m\)).

**B. Passive Time Reversal with Source-Array Range Shift**

If there is a source-array range shift \(\Delta r = r - R\) (where \(r\) is the new range) between the first and the second transmissions the resulting pTR pressure field is given by

\[
P_{pc}(\cdot; \Delta r) = \sum_{i} G_{\omega}(R + \Delta r, z_0, z_i) G_{\omega}^*(R, z_0, z_i)
\]

\[
= \frac{1}{\rho^2 8\pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|} e^{j(k_m (R + \Delta r) - k_m R)},
\]

(32)

\(^2\text{i.e., that there is a sufficiently large number of hydrophones, the vertical array is spanning the whole water column and the propagation environment is time-invariant.}\)
where $\Delta r$ is small enough in order to be considered negligible in the denominator. The argument of the complex exponential in (32) is no longer null and this is responsible for a loss of performance of pTR.

Song et al [10] developed a waveguide invariant based method [11] to perform range shift of the aTR focal spot whose application to pTR range shift compensation is straightforward. As originally proposed this method only emphasizes the pressure field intensity, and a different interpretation of the compensation mechanism is given here to account for the phase effect over the frequency domain pressure field, that results in a delay in the time domain. According to [10] the source-array range distortion over the pTR can be compensated by applying a frequency shift $\Delta \omega$ to the first received pressure field (index $m$) in (32), i.e.,

$$G_{\omega+\Delta \omega}(R, z_0, z_i) = \frac{-j}{\rho \sqrt{8\pi R}} e^{-j \frac{\pi}{4}} \sum_{m=1}^{M} Z_m(z_i) Z_m(z_0) e^{ik_m(\omega+\Delta \omega)R},$$  \hspace{1cm} (33)

where under the adiabatic condition the only quantity that is considered to be frequency dependent is the horizontal wavenumber $k_m$ in the argument of the complex exponential (the frequency shift in the $k_m$ placed in the denominator is negligible). Using (33) in (32) results in

$$P_{pe}(\cdot; \Delta r, \Delta \omega) = \sum_i G_{\omega}(R + \Delta r, z_0, z_i) G^*_{\omega+\Delta \omega}(R, z_0, z_i)$$

$$= \frac{1}{\rho^2 8\pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|} e^{j(k_m(R+\Delta r)-k_m^1 R)},$$  \hspace{1cm} (34)

where $k_m^1 = k_m(\omega + \Delta \omega)$. Considering the Taylor first order approximation for the frequency shift horizontal wavenumber

$$k_m^1 = k_m + \frac{dk_m}{d\omega} \Delta \omega$$

$$= k_m + \frac{1}{u_m(\omega)} \Delta \omega,$$  \hspace{1cm} (35)

where $u_m$ is the horizontal group velocity. Replacing (35) in (34) yields

$$P_{pe}(\cdot; \Delta r, \Delta \omega) = \frac{1}{\rho^2 8\pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|} e^{j k_m \Delta r} e^{-j \frac{d k_m}{d\omega} \Delta \omega R}.$$  \hspace{1cm} (36)
The frequency shift $\Delta \omega$ used in [10] to perform a range shift of the aTR focal spot can be adapted here to compensate for the pTR range shift $\Delta r$, and is given by

$$\Delta \omega = -\frac{\omega}{R} \Delta r \beta.$$  \hfill (37)

where $\beta$ is the horizontal waveguide invariant that can be chosen from a set of possible values (8) or given by (12). In the following the second option will be considered.

As previously mentioned, in [10] only the aTR focal spot intensity was considered, and no attempt was made to understand the range shift influence over the aTR focal spot phase. When applying the same strategy to compensate for pTR range mismatch, namely for underwater communications, the argument of the exponential term in (34) should be approximately linear with frequency and range shift (to avoid signal distortion). That is made possible by using (11) and (37) in the exponential term (36)

$$k_m \Delta r + \frac{d k_m}{d \omega} \omega \beta_e \Delta r \approx \omega \rho_{\beta_e} \Delta r,$$  \hfill (38)

where $\beta_e$ and $\rho_{\beta_e}$ are given by (12) and (13) respectively. Finally replacing (38) in (36) results in

$$P_{pc}(:, \Delta r, \Delta \omega) \approx \frac{e^{j \rho_{\beta_e} \omega \Delta r}}{\rho^2 8\pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|^2}.$$ \hfill (39)

When compared with (31), it is clear that, except for a harmless linear phase term, the source-array range mismatch has been compensated in (39).
C. Passive Time Reversal with Source Depth Shift

If there is a source depth shift $\Delta z_0$ between the first and the second transmissions the pTR pressure field becomes

$$P_{pc}(\cdot; \Delta z_0) = \sum_i G_\omega(R, z_0 + \Delta z_0, z_i)G^*_\omega(R, z_0, z_i)$$

$$= \frac{1}{8\pi R} \sum_{m=1}^M \sum_{n=1}^M \frac{Z_m(z_0)Z_n(z_0 + \Delta z_0)}{\rho} \sum_i \frac{Z_n(z_i)Z_m(z_i)}{\rho} \frac{e^{j(k_mR - k_nR)}}{\sqrt{k_mk_n}}$$

$$= \frac{1}{\rho8\pi R} \sum_{m=1}^M \left[ \frac{Z_m(z_0)Z_m(z_0 + \Delta z_0)}{\rho} \right] \frac{1}{|k_m|^4},$$

(40)

where the term in $[\cdot]$ is responsible for a loss of performance of pTR.

In a perfect waveguide with no source depth mismatch, $\rho = 1$, and considering the full set of modes,

$$\sum_{m=1}^M \frac{Z_m(z_0)Z_m(z)}{\rho} \approx \begin{cases} \frac{D}{2} & \text{if } z = z_0 \\ 0 & \text{if } z \neq z_0 \end{cases},$$

(41)

which means that even with a small depth mismatch $\Delta z_0 = z - z_0$ the pTR performance strongly degrades, since (41) results in a weighted dirac in depth $^3$.

Applying to pTR the basic idea of Walker [12] for aTR focal spot depth shift, the term in $[\cdot]$ can be written as

$$\sum_{m=1}^M \left[ \frac{Z_m(z_0)Z_m(z_0 + \Delta z_0)}{\rho} \right] = \sum_{m=1}^M \left[ \frac{Z_m(z_0)Z_m(z_0)}{\rho} \left\{ \frac{Z_m(z_0 + \Delta z_0)}{Z_m(z_0)} \right\} \right].$$

(42)

This implies that a compensation of the source depth shift can be achieved by applying the inverse of the term in $\{\cdot\}$ to (40). The method proposed in [12] requires the explicit computation of $Z_m(z)$ using, e.g., the data-based mode extraction method proposed in [19] that requires the collection of data at several ranges. Although the basic idea is the same, the compensation method presented here does not require the explicit computation of $Z_m$;

$^3$It should be mentioned that in a real scenario the weighted dirac is replaced by a pulse with a main lobe, that originates a slower performance degradation with the depth mismatch, when compared with the theoretical case.
instead, by using a frequency shift of the acoustic field, an approximation to the inverse of the term in \{\cdot\} is computed implicitly.

It was found that in a perfect waveguide the source depth mismatch compensation can be partially achieved multiplying by \(\cos(\gamma_m \Delta z_0)\) the left term of (42)\(^4\), that is

\[
\sum_{m=1}^{M} Z_m(z_0)Z_m(z_0 + \Delta z_0)\cos(\gamma_m \Delta z_0).
\]  

(43)

For the perfect waveguide mode shape (43) can be rewritten as

\[
\frac{D}{2} \sum_{m=1}^{M} \sin(\gamma_m z_0) \sin(\gamma_m z_0 + \gamma_m \Delta z_0) \cos(\gamma_m \Delta z_0),
\]  

(44)

that, by trivial trigonometric manipulation, becomes

\[
\frac{D}{2} \sum_{m=1}^{M} \frac{1}{4} [1 + \cos(2\gamma_m \Delta z_0) - \cos(2\gamma_m z_0) - \cos(2\gamma_m z_0 + 2\gamma_m \Delta z_0)].
\]  

(45)

The sum of oscillating terms in (45) is approximately zero except for \(\Delta z_0 = 0\) due to \(\cos(2\gamma_m \Delta z_0)\), and the term 1 is responsible for the compensation. It results that

\[
\sum_{m=1}^{M} Z_m(z_0)Z_m(z_0 + \Delta z_0)\cos(\gamma_m \Delta z_0) \approx \left\{ \begin{array} {c} \frac{D}{2} \frac{M}{4} \text{ if } \Delta z_0 = 0 \\ \frac{D}{2} \frac{M}{4} \text{ if } \Delta z_0 \neq 0 \end{array} \right..
\]  

(46)

Comparing (41) and (46) it is clear that under depth mismatch the compensation can restore half the magnitude of (41). Similarly \(\exp(j\gamma_m \Delta z_0)\) (the negative exponential is also a viable choice) can be used for compensation instead of \(\cos(\gamma_m \Delta z_0)\), in which case one must consider an additional imaginary component that will be responsible for the presence of a linear phase\(^5\)

with \(\Delta z_0\)

\[
\sum_{m=1}^{M} Z_m(z_0)Z_m(z_0 + \Delta z_0)e^{j\gamma_m \Delta z_0} \approx \sum_{m=1}^{M} Z_m(z_0)Z_m(z_0)\frac{V(\Delta z_0)}{2},
\]  

(47)

where \(|V(\Delta z_0)| \approx 2\) for \(\Delta z_0 = 0\) and \(|V(\Delta z_0)| \approx 1\) for \(\Delta z_0 \neq 0\).

\(^4\)the choice of \(\cos(\gamma_m \Delta z_0)\) was made in aTR context and further adapted to pTR by considering an analogy with the demodulation of a single-side-band modulation using a double-side-band demodulator.

\(^5\)in a real scenario due to the previously mentioned replacement of the weighted dirac in (41) by a pulse with a main lobe, it results that the linear phase with \(\Delta z_0\) becomes a dual-slope linear phase (one inside the lobe and the other outside), that can be observed in Figure 4 for simulated data.
In [1] it is considered that for aTR (41) can be applied approximately by assuming that the \( k_m \)'s are nearly constant over the interval of the contributing modes and can be replaced by their mean \( |k_m| \). Here, in a similar manner (47) is used for the compensated pTR and results in the source depth shift compensated pTR pressure field given by

\[
P_{pc}^{comp}(\cdot; \Delta z_0) \approx \frac{1}{\rho^2 8 \pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2 V(\Delta z_0)}{|k_m|}. \tag{48}
\]

As previously, a frequency shift is used to implicitly perform the compensation based on the product by \( \exp(j \gamma_m \Delta z_0) \). Considering the approximation to the vertical wavenumber using the horizontal group slowness (22), in a similar manner to the compensation of the source-array range mismatch (38) one can write

\[
\gamma_m \Delta z_0 \approx -\frac{d}{d\omega} \omega \zeta_0 \Delta z_0 + \rho \zeta, 0 \omega \Delta z_0, \tag{49}
\]

where \( \zeta_0 \) is given by (23) and \( \rho \zeta, 0 \) is given by (24). Taking

\[
\Delta \omega = \frac{\omega}{R} \Delta z_0 \zeta_0, \tag{50}
\]

in the first-order Taylor expansion (35), results in the compensated field

\[
P_{pc}(\cdot; \Delta z_0, \Delta \omega) = \sum_i G_\omega(R, z_0 + \Delta z_0, z_i) G_\omega^*(R, z_0, z_i)
\]

\[
= \frac{1}{\rho^2 8 \pi R} \sum_{m=1}^{M} \frac{Z_m(z_0 + \Delta z_0) Z_m(z_0) e^{-j \frac{dk_m}{d\omega} \Delta \omega R}}{|k_m|} 
\]

\[
\approx \frac{V(\Delta z_0)}{2} e^{-j \rho \zeta, 0 \omega \Delta z_0} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|}. \tag{51}
\]

When compared with (31) it is clear that, at the cost of an attenuation and a harmless linear phase term the source depth mismatch pTR acoustic field becomes partially compensated.
D. Passive Time Reversal with Array Depth Shift

If there is an array depth shift $\Delta z_i$ between the first and the second transmissions the mismatched pTR pressure field becomes

$$P_{pc}(\cdot; \Delta z_i) = \sum_{i} G_\omega (R, z_0, z_i + \Delta z_i) G_\omega^*(R, z_0, z_i)$$

$$= \frac{1}{\rho^2 8 \pi R} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{Z_m(z_0) Z_n(z_0)}{\sqrt{k_m k_n}} \sum_i Z_n(z_i + \Delta z_i) Z_m(z_i) e^{i(k_m R - k_n R)}$$

$$= \frac{1}{\rho^2 8 \pi R} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{Z_m(z_0) Z_n(z_0)}{\sqrt{k_m k_n}} \Psi(m, n, \Delta z_i) e^{i(k_m R - k_n R)}. \quad (52)$$

Unlike $\Psi(m, n)$ in (29) and (30) now $\Psi(m, n, \Delta z_i)$, is no longer an impulse and the two summations $\sum_{m=1}^{M} \sum_{n=1}^{M} (\cdot)$ cannot be replaced by a single one. Instead they must be replaced by $\sum_{m=n}^{M} (\cdot) + \sum_{m \neq n}^{M} (\cdot)$, where the cross terms will be responsible for a loss of performance of the pTR processor.

Appendix B studies the effect of the array depth mismatch over the mode orthogonality property, and proposes a compensation strategy that partially recovers it in presence of array depth mismatch. It results that partial compensation can be achieved using (B16) in (52)

$$P_{pc}^{comp}(\cdot; \Delta z_i) \approx \frac{1}{\rho^2 8 \pi R} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{|Z_m(z_0)|^2 |Z_n(z_0)|}{|k_m|} \Psi(m, n, \Delta z_i) e^{i(k_m R - k_n R)}.$$

The summation $\sum_{m=1}^{M} \sum_{n=1}^{M} (\cdot)$ can now be replaced by $\sum_{m=n}^{M} [W(m, \Delta z_i), (\cdot)]$, and the resulting pTR pressure field will be given by

$$P_{pc}^{comp}(\cdot; \Delta z_i) \approx \frac{1}{\rho^2 8 \pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2 W(m, \Delta z_i)}{|k_m|}. \quad (54)$$

As in (48) a frequency shift implicitly introduces the required factor $exp(j\gamma_m \Delta z_i)$ (note that $exp(-j\gamma_m \Delta z_i)$ is also a viable choice since in appendix B the compensation is originally attained by $cos(\gamma_m \Delta z_i))$. Considering the approximation to the vertical wavenumber using
the horizontal group slowness \( \gamma_m \Delta z_0 \approx -\frac{dk_m}{d\omega} \omega \zeta_i \Delta z_i + \rho_{\zeta,i} \omega \Delta z_i \),

\[ (55) \]

where \( \zeta_i \) is given by (23) and \( \rho_{\zeta,i} \) is given by (24). Considering the Taylor first order expansion (35) and

\[ \Delta \omega = \frac{\omega}{R} \Delta z_i \zeta_i, \]

\[ (56) \]

yields the compensated field

\[
P_{pc}(\cdot; \Delta z_i, \Delta \omega) = \sum_i G_\omega(R, z_0, z_i + \Delta z_i)G^{*}_{\omega + \Delta \omega}(R, z_0, z_i)
\]

\[
= \frac{1}{\rho^2 \pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|} \Psi(m, n, \Delta z_i)e^{-j \frac{2\pi}{\omega} \Delta \omega R}
\]

\[
\approx \frac{e^{-j \rho \omega \Delta z_i}}{\rho^2 \pi R} \sum_{m=1}^{M} \frac{|Z_m(z_0)|^2}{|k_m|} \frac{W(m, \Delta z_i)}{2}, \]

\[ (57) \]

that resembles (51), and similar comments apply.

IV. Simulations with a perfect waveguide

In order to illustrate the behavior of the geometric mismatch frequency shift compensation mechanism, simulations were conducted for a perfect waveguide with \( D = 100 \) m water column depth, \( c = 1500 \) m/s, a frequency \( f = 1 \) kHz, a nominal source-array range \( R = 1 \) km, a nominal source depth \( z_0 = 50 \) m and a vertical line array of 81 hydrophones, 1 m spaced, centered in the water column.

The results for the source-array range mismatch compensation given by the pTR acoustic field \( P_{PC}(\cdot; \Delta r, \Delta \omega) \) of (34), source depth mismatch compensation given by \( P_{PC}(\cdot; \Delta z_0, \Delta \omega) \) of (51), and array depth mismatch compensation given by \( P_{PC}(\cdot; \Delta z_i, \Delta \omega) \) of (57), are shown in Figures 3, 4 and 5, respectively. The dashed line shows the behavior of pTR under the geometric mismatch \( \Delta (\Delta r, \Delta z_0, \Delta z_i) \) with no compensation, \( \Delta \omega = 0 \) where \( \Delta \omega = 2\pi \Delta f \). The continuous line shows the maxima of the surface \( |P_{PC}(\cdot; \Delta, \Delta \omega)| \) in (a) and the respective
phase in (b). The dot-dashed line represents the pTR acoustic field with the proposed compensation mechanism given by (37), (50) and (56) for source-array range, source depth and array depth mismatches, respectively. In all three cases the waveguide invariants $\beta_e$, $\zeta_{e,0}$, $\zeta_{e,i}$, $\rho_{\beta,e}$, $\rho_{\zeta,e,0}$ and $\rho_{\zeta,e,i}$ were computed for an effective number of modes $M_e = M/2$.

Figure 3(a) shows the pTR acoustic field magnitude $|P_{PC}(\cdot; \Delta r, \Delta \omega)|$ of (34) where it is clear that the compensation $\Delta \omega$ behaves linearly with the mismatch $\Delta r$ with a slope invariant $\beta_e \approx 0.88$ given by (12). For each $\Delta r$ the frequency shift for which the maximum magnitude is attained almost equals (37), and provides a strong gain when compared with the no compensation case given by the dashed line. Figure 3(b) shows the unwrapped phase of (36) for the three cases described before. Except for the $\max(P_{PC})$ case the others exhibit a linear trend with $\Delta r$ that, for the proposed frequency shift compensation mechanism, is controlled by $\rho_{\beta,e}$ given by (13).

![Figure 3](image)

Figure 3: Frequency shift source-array range mismatch compensation given by the pTR acoustic field $P_{PC}(\cdot; \Delta r, \Delta \omega)$ of (34); (a) normalized magnitude, (b) unwrapped phase.

Figure 4(a) shows the magnitude of (51) where an almost symmetric behavior with $\Delta f$ is observed since the compensation can be done either with $\exp(j \gamma_m \Delta z_0)$ or $\exp(-j \gamma_m \Delta z_0)$ (recall that compensation is originally attained using $\cos(\gamma_m \Delta z_0)$ and that the positive and the negative exponentials are both easy to implement alternative choices). This near sym-
metry makes the maximum to toggle between positive and negative values of the frequency shift. Considering only one side of the symmetry it can be observed that the compensation \( \Delta \omega \) behaves almost linearly with the mismatch \( \Delta z_0 \) with a slope invariant \( \zeta_{e,0} \approx 3.14 \) given by (23) (such linearity becomes a weaker approximation for high frequencies and high source depth mismatches). Figure 4(a) shows that the frequency shift compensation maximum and the frequency shift given by (50) attain a stronger magnitude than the non-compensated case given by the dashed line when the main lobe, which can be seen in center of the figure, reaches one half of the maximum value given for \( \Delta z_0 = 0 \) as predicted by (46). The existence of the pTR main lobe, whose main lobe size increases when \( M_e \) decreases, was not predicted in the theoretic development since the approximations (41) and (46) were directly used in (48). This fact, together with the approximation (49), are also responsible for the performance loss of the compensation mechanism that can be observed in the figure as \( \Delta z_0 \) increases.

Figure 4: Frequency shift source depth mismatch compensation given by the pTR acoustic field \( P_{PC}(\cdot; \Delta z_0, \Delta \omega) \) of (51); (a) normalized magnitude, (b) unwrapped phase.

Figure 4(b) shows the phase of (51) for the three cases described above. An almost linear trend with \( \Delta z_0 \) is observed only for the proposed compensation mechanism that is controlled by \( \rho_{\zeta,e} \) given by (24) and the phase of \( V(\Delta z_0) \) (in fact a detailed observation shows a tenuous
two slope linear phase with one slope inside the \( |P_{PC}(\cdot; \Delta z_0, \Delta \omega)| \) main lobe and a different one outside).

Figure 5 shows the simulation behavior of the array depth mismatch frequency shift compensation mechanism for a perfect waveguide similar to the one used for Figures 3 and 4. The behavior is quite similar to the one observed for the source depth mismatch compensation case and similar comments apply, with the main difference being that now function \( W(m, \Delta z_i) \) plays the role of \( V(\Delta z_0) \). Note that the slope invariant \( \zeta_{e,i} = \zeta_{e,0} \approx 3.14 \) are both given by (23) and are equal due to the fact that the Sound Speed Profile (SSP) is constant over the water column, which makes the vertical wavenumber \( \gamma_m \) depth independent. Additional comments when in presence of a depth dependent SSP will be given in Sections V. and VI.

(a) \hspace{1cm} (b)

Figure 5: Frequency shift array depth mismatch compensation given by the pTR acoustic field \( P_{PC}(\cdot; \Delta z_i, \Delta \omega) \) of (57); (a) normalized magnitude, (b) unwrapped phase.

V. Extension of the geometric compensation mechanisms to realistic waveguides

The intrinsic behavior of the geometric mismatch compensation mechanism in a perfect waveguide can be further extended to more realistic waveguides by considering the Pekeris
waveguide and the WKB approximation. In a Pekeris waveguide the high order modes become leaky due to the presence of the half space and at sufficiently larger range they have no contribution to the received acoustic field [20]. That means that in a real waveguide the mode limitation required to improve the \( k_m' \) to \( k_m \) approximation (11) and \( \gamma_m' \) to \( \gamma_m \) approximation (22) is performed up to a certain point by the waveguide itself.

Although the WKB approximation is usually applied in deep water, some insight into the behavior of the modes in a waveguide with a depth-dependent SSP can be obtained. In [21] it is shown that under the WKB approximation the group slowness is given by

\[
\frac{dk_m}{d\omega} = \frac{\omega}{k_m} \int \frac{1}{\rho(z)c(z)} Z_m^2(z) dz \int \frac{1}{\rho(z)\sigma_m^2(z)} dz
\]

(58)

where \( \rho(z) \) and \( c(z) \) are the depth-dependent water density and SSP, respectively, and the ratio of integrals in the right hand side is a constant. Using the WKB horizontal group slowness (58) instead of the perfect waveguide horizontal group slowness (5) the approximations (11) and (22) are still valid. Heuristically, this suggests that the genesis of the range shift compensation mechanism under the WKB approximation with a depth-dependent SSP remains unchanged.

Regarding the use of \( \exp(\pm j\gamma_m \Delta z) \) to compensate for the source depth shift and array depth shift, the following should be considered. The WKB approximation reveals that in the presence of a ducted SSP [17, 21] the mode shape is no longer ruled by a constant vertical wavenumber, \( \gamma_m \), but by a depth-dependent \( \gamma_m(z) \). Under the WKB approximation the mode shape between the points where it vanishes, that is, between the ray turning points in the ray mode analogy, is given by

\[
Z_m(z) = A_m(z) \sin \Upsilon_m(z),
\]

(59)

where \( A_m(z) \) is a slowly-varying amplitude

\[
A_m(z) = \gamma_m^{-1/2}(z) = \left[ \frac{\omega^2}{c^2(z)} - k_m^2 \right]^{-1/4},
\]

(60)
the WKB phase is
\[ \Upsilon_m(z) = \Upsilon_a + \int_a^z \gamma_m(z')dz', \] (61)
where \( a \) is the upper mode vanishing depth (or the equivalent ray turning depth) and the depth dependent vertical wavenumber is
\[ \gamma_m^2(z) = \frac{\omega^2}{c^2(z)} - k_m^2, \] (62)
where \( k_m \) is the depth-independent horizontal wavenumber. In presence of a depth shift \( \Delta z \) one can compute the new vertical wavenumber \( \gamma_m(z + \Delta z) \) using the nominal \( \gamma_m(z) \) and the sound speed at \( z \) and \( z + \Delta z \)
\[ \gamma_m^2(z + \Delta z) = \frac{\omega^2}{c^2(z)} \sigma_c(z, \Delta z) + \gamma_m^2(z), \] (63)
where
\[ \sigma_c(z, \Delta z) = \frac{c^2(z) - c^2(z + \Delta z)}{c^2(z + \Delta z)}. \] (64)
Replacing (62) in (63) results in
\[ \gamma_m^2(z + \Delta z) = \frac{\omega^2}{c^2(z)}(\sigma_c(z, \Delta z) + 1) - k_m^2, \] (65)
where \( \sigma_c(z, \Delta z) \approx 0 \) and \( \gamma_m^2(z + \Delta z) \approx \gamma_m^2(z) \) if the sound speed varies slowly with depth.
In the presence of a depth shift \( \Delta z \) the WKB phase (61) becomes
\[ \Upsilon_m(z + \Delta z) = \Upsilon_a + \int_a^z \gamma_m(z')dz' + \int_z^{z+\Delta z} \gamma_m(z')dz', \] (66)
and, since \( \sigma_c(z, \Delta z) + 1 \approx 1 \) for a slowly-varying SSP, with small \( \Delta z \), the third term of (66) becomes
\[ \int_z^{z+\Delta z} \gamma_m(z')dz' \approx \gamma_m(z)\Delta z, \] (67)
and the mode shape \( Z_m(z) \) becomes
\[ Z_m(z + \Delta z) \approx A_m(z) \sin[\Upsilon_m(z) + \gamma_m(z)\Delta z]. \] (68)
Note the similarity to the mode shape in a perfect waveguide, given by
\[ Z_m(z + \Delta z) = \sin(\gamma_m z + \gamma_m \Delta z). \]

Considering now that a real waveguide is a Pekeris waveguide with a WKB approximation such that the mode shape in the water column is given by the WKB approximation, but near the boundaries the mode vanishing locations (ray turning points) are given by the boundaries of the waveguide as in the Pekeris waveguide, the \( \Psi(m, n, \Delta z_i) \) function of (B5) becomes
\[
\Psi(m, n, \Delta z_i) = \Xi \int_0^D \sin[\Upsilon_m(z) + \gamma_m(z)\Delta z_i)] \sin(\Upsilon_n(z))dz, \quad (69)
\]
where the assumed constant \( \Xi \) is a function of water depth \( D \) and \( A_m(z) \) of (60). The later is a slowly-varying function of \( z \) and assumed constant within the water column.

Comparing (B5) for the perfect waveguide with (69) for the more realistic cased based on the Pekeris waveguide with the WKB approximation, one can see that the distortion is similar, the main difference being that for (69) the vertical wavenumber is depth dependent. Such dependence suggests that a short array placed in the middle of the water column must be used in order to consider \( \gamma_m(z) \approx \gamma_m(z) \), or that for large vertical arrays the compensation should be applied in small array sections. For the proposed pTR array depth mismatch compensation mechanism applying the compensation in small sections of the array is not a problem since the pTR is a linear operator over the array.

VI. Geometric mismatch compensation with experimental data

The experimental data were collected during the MREA’04 sea trial that took place off the town of Setúbal, approximately 50km from Lisbon (Portugal) in April 2004. The acoustic source was suspended from the NRV Alliance at a nominal depth of 70m, the receiving VLA was surface-suspended from the free-drifting Acoustic Oceanographic Buoy (AOB) [22] and comprised two sections. The upper section with two hydrophones at nominal depths of 10
and 15m and the lower section with six hydrophones at nominal depths of 55, 60, 65, 70, 75 and 80m (in the following only the lower section will be used). The pTR experiment data processed in this section were acquired at a close range between 0.6 and 0.8km to the south of the acoustic source in a constant up slope region with a water column depth varying between 90 and 110m, at an almost constant source-array relative speed of about 0.6m/s. The environment was characterized by a thermocline of approximately 20m over a downward refracting SSP over a 1.5m silt bottom and a gravel sub-bottom. The surface-suspended AOB was affected by a wave height of approximately 0.63m with frequency between 0.43 and 0.4Hz, as measured with a coastal monitoring buoy placed in the area of the experiment. The source depth was measured at a sampling rate of 1 second with 10cm resolution and oscillates between 71.64 and 72.24m. The power spectral density of the source depth data reveals the presence of one main component at 0.1Hz.

The transmission set processed here corresponds to modulated data at a carrier frequency of 3.6kHz, using a 400 baud symbol rate with 2-PSK constellation and fourth-root raised cosine signaling pulses with 100% roll-off, such that the signal bandwidth is 800Hz. Each individual transmission comprises a single PAM signaling pulse acting as a channel probe with symmetric guard intervals and a total duration of 1s, followed by a 20s data packet. The source transmits four packets with a total duration of 84s.

Before any geometric mismatch compensation the data were synchronized, Doppler compensated and channel identification was performed independently for each hydrophone using the exponentially-windowed RLS algorithm (see [23] and [24] for details). Each data packet provides a set of 400 IR estimates at intervals of 0.05s, where the first 9 estimates were rejected. The proposed pTR geometric mismatch compensation is then applied to 78.2s of data (collected during 83s) that corresponds to 1564 IR estimates with the IR estimate number 782 being used as the nominal IR. This results in a setup where the source-array range mismatch $\Delta r = r - R$ starts at negative values and then gradually increases up to the
1564 IR estimate.

Figure 6 shows the arriving pattern of the lower sections of the array for the middle IR estimate where four relevant paths are clearly identified.

![Figure 6: Arrival pattern of the non mismatch IR estimate.](image)

Applying the proposed frequency shift geometric mismatch compensation strategy to the pTR operator, with only the first two paths of Figure 6, the surface of Figure 7 is obtained. Since the range mismatch changes from negative to positive values, crossing zero at about 40 s (that corresponds to the nominal IR), it is expected that the frequency shift which compensates for the range mismatch varies from positive to negative values as in Figure 3(a) for the simulation in a perfect waveguide. This is confirmed in Figure 7 that shows a patch of pTR maxima ranging from a frequency shift mean value of about −90 up to 90 Hz. The dashed line represents the expected frequency shift obtained with (37) by considering the range shift obtained from the NRV Alliance and AOB real-time GPS data, the data carrier frequency \( f_c = 3.6 \text{kHz} \) and a horizontal waveguide invariant \( \beta = 0.81 \). The \( \beta \) value was computed from an approximation for constant slope range-dependent environments with constant sound speed derived by D’Spain [25]. Originally, \( \beta \) was proposed to be the ratio between the source and the array position water-column depths, \( \beta \approx D(S)/D(VLA) \). When
applied in the present context (with the compensation mechanism applied in the reversed field) it should be considered that pTR implements a synthetic aTR with a virtual transmitting/receiving array at the VLA location and a virtual receiver at the source location. It results that the $\beta$ invariant as a function of range is approximately given by

$$\beta \approx \frac{D(VLA)}{D(S)}.$$  \hfill (70)

Although the dashed line is just an approximation, Figure 7 confirms that (37) produces a good fit to the actual path of maximum magnitude (solid line). This similarity suggests that the frequency shift compensation is dominated by the range mismatch.

Figure 7: The surface represents the normalized magnitude of $P_{pc}(\cdot; \Delta, \Delta \omega)$, when the nominal IR is at $t = 40s$ and considering the IRs limited to two arriving paths. The solid line traces the maxima of the surface over time. The dashed line represents the expected behavior if there is only range mismatch.

Figure 8 shows the power spectral density of the slice of $P_{pc}(\cdot; \Delta, \Delta \omega)$ along the maximum line of Figure 7, where three main components are identifiable at approximately 0.21, 0.3 and 0.4Hz (the first maximum at 0.012Hz is due to the 83s data packet length). Superimposed on the $P_{pc}(\cdot; \Delta, \Delta \omega)$ along the maximum line power spectrum, with ‘*’, one can see the power spectrum of the source depth time series during the same data packet. Both curves display a similar behavior, despite a displacement in frequency. This fact, together with the array
depth oscillation induced by the 0.4Hz surface waves allow us to speculate that the swing of the $P_{pc}(\cdot; \Delta, \Delta\omega)$ along the maximum line curve in Figure 7 is due to source and array depth oscillations. Nevertheless, no definitive conclusions can be attained since no accurate \textit{in situ} source and array depth measurements with fast enough sampling rate are available.

Figure 8: Power spectrum of the slice of $P_{pc}$ along the maximum line of Figure 7 (solid line), and power spectrum of the source depth time series (‘*’).

It is interesting to observe that the symmetric ducts of the source and array depth compensations in the ideal waveguide of Figures 4(a) and 5(a) vanish in Figure 7. That is due to the presence of a depth-dependent SSP that causes the vertical wavenumber $\gamma_m(z)$ to become depth dependent as well.

Figure 9 is similar to Figure 7, but was computed with the first three paths of Figure 6. The main difference between these two figures is the appearance of two more $P_{PC}$ maximum patches almost parallel to the original one. Those patches do not have the same nature as the symmetric patches of the source and array depth compensations in an ideal waveguide (see Figures 4 and 5), those patches originate from aliasing that are due to poor spatial sampling of the high-order modes in the 5m spaced hydrophone array.
VII. Conclusions and future work

An analytical model based waveguide invariant approach has been developed for pTR geometric mismatch compensation in shallow water. It was found that the horizontal waveguide invariant $\beta$ can be used to approximate the horizontal wavenumber using the group slowness, and that a new vertical waveguide invariant $\zeta$ can be used to approximate the vertical wavenumber. By analogy with the invariance of $\beta$ in the frequency/range plane, $\zeta$ would be invariant in the frequency/depth plane. After theoretically establishing the effect of geometric mismatch over the pTR operation for a perfect waveguide, it was found that an appropriate frequency shift calculated using the invariants can then be used for compensation during the pTR operation. The compensation method was extended heuristically to realistic waveguides and its usefulness was shown with real data.

In the present paper the invariance of $\zeta$ in the frequency / depth plane was not completely demonstrated, mainly due to the absence of appropriate real data. Future experiments should be design in order to overcome such problem, by providing accurate measurements of the source and array depths.

The proposed compensation method is potentially useful, e.g., in underwater communi-
cation systems based on pTR, where the presence of uncompensated geometric mismatch between the probe-signal and actual data transmissions degrades the performance. In fact, an underwater communication system, based on the developed geometric mismatch compensation, is proposed in [14]. It results in a environmental equalizer that uses a relatively small number of coefficients to deconvolve the cannel multipath, unlike most current equalizers.

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A Linear approximation of monotonic functions

Sections C. and D. require the computation of horizontal and vertical wavenumbers using the horizontal wavenumber inverse. Both problems can be seen as a generic linear approximation of one monotonic function, \( \Phi_m \), using another monotonic function, \( \Pi_m \).

Considering the linear approximation of \( \Phi_m \) using \( \Pi_m \) with \( m = 1 \ldots M \) in the least-squares sense, it results

\[
\Phi_m \approx \Phi'_m = -\varepsilon \Pi_m + \rho,
\]

where

\[
\varepsilon = -\frac{\Phi_m \Pi_m - \bar{\Phi}_m \bar{\Pi}_m}{\bar{\Pi}_m^2 - \bar{\Pi}_m^2},
\]

and

\[
\rho = \bar{\Phi}_m + \varepsilon \bar{\Pi}_m,
\]

where the bar denotes the mean over \( m \). If both functions \( \Phi_m \) and \( \Pi_m \) are linear with
When one or both functions are non-linear with different curvatures the approximation will have an error that can be reduced if instead of approximating $\Phi_m$ for $m = 1..M$ only a subset of $m$ is considered. For the approximation of horizontal and vertical wavenumbers an effective number of modes $M_e$ smaller than the total number of modes $M$ will be considered. That will result in a linear approximation given by

the parameters $\varepsilon_e$ and $\rho_e$.

A different approximation to $\Phi_m$ using $\Pi_m$ is given by computing

$$
\Phi'_m = -\varepsilon_{\mu, \nu} \Pi_m + \rho_{\mu, \nu},
$$

with

$$
\varepsilon_{\mu, \nu} = -\frac{\Phi_{\mu} - \Phi_{\nu}}{\Pi_{\mu} - \Pi_{\nu}}, \quad (A4)
$$

and

$$
\rho_{\mu, \nu} = \Phi_{\nu} + \varepsilon_{\mu, \nu} \Pi_{\nu}, \quad (A5)
$$

where $m = \nu$ and $m = \mu$ are the abscissa for $\Phi_{\nu} = \Phi'_\nu$ and $\Phi_{\mu} = \Phi'_\mu$. It results that $\Phi'_m \approx \Phi_m$ with different degrees of accuracy given by the selected $\nu$ and $\mu$.

Since (A4) represents a set of linear approximations with the only constraint that the two functions $\Phi_m$ and $\Phi'_m$ meet at two different points $m = \nu$ and $m = \mu$, it is expected that the linear least-squares approximation (A1) will coincide or be close to one of them, i.e, for each $M_e$ there is a $(\mu, \nu)$ such that $\varepsilon_e \approx \varepsilon_{\mu, \nu}$ and $\rho_e \approx \rho_{\mu, \nu}$.

**B  Mode orthogonality in the presence of an array depth mismatch**

When there is no array depth mismatch the mode orthogonality condition is given by

$$
\Psi(m, n) = \int_0^D \frac{Z_m(z)Z_n(z)}{\rho(z)} dz = \delta_{m,n}, \quad (B1)
$$
where $D$ is the waveguide water column depth, $\rho(z)$ is the water density consider to be constant and equal to 1, and $z$ is the depth. The mode shape $Z_m(z)$ in a perfect wave guide is given by

$$Z_m(z) = \sqrt{\frac{D}{2}} \sin(\gamma_m z), \quad (B2)$$

with

$$\gamma_m = \left( m - \frac{1}{2} \right) \frac{\pi}{D}. \quad (B3)$$

Using (B2) in (B1) it results

$$\Psi(m, n) = \frac{2}{D} \int_0^D \sin(\gamma_m z) \sin(\gamma_n z) dz, \quad (B4)$$

and (B1) follows readily.

If now in (B4) there is a depth shift between the mode functions $Z_m$ and $Z_n$, it results

$$\Psi(m, n, \Delta z) = \frac{2}{D} \int_0^D \sin(\gamma_m z - \gamma_m \Delta z) \sin(\gamma_n z) dz. \quad (B5)$$

Using Euler formula, and ignoring the backward propagating modes (with $m$ and $n$ negative integers), (B5) becomes

$$\Psi(m, n, \Delta z) \approx -\frac{2}{4D} \left[ \int_0^D \Lambda'_{m,n} dz + \int_0^D \Omega'_{m,n} dz \right], \quad (B6)$$

where

$$\Lambda'_{m,n} = \Lambda_{m,n} \exp(-j\pi(m - 1/2)\Delta z/D),$$
$$\Omega'_{m,n} = \Omega_{m,n} \exp(j\pi(m - 1/2)\Delta z/D), \quad (B7)$$

and

$$\Lambda_{m,n} = -\exp(j\pi(m - n)z/D),$$
$$\Omega_{m,n} = -\exp(-j\pi(m - n)z/D). \quad (B8)$$
Defining
\[
\Psi_A(m, n) = \int_0^D \Lambda_{m,n} dz, \\
\Psi_\Omega(m, n) = \int_0^D \Omega_{m,n} dz, \tag{B9}
\]
using (B3) and the fact that the exponential terms in (B7) do not depend on \( z \), (B6) becomes
\[
\Psi(m, n, \Delta z) = -\frac{2}{4D} \left[ \exp(-j\gamma m \Delta z) \Psi_A(m, n) + \exp(j\gamma m \Delta z) \Psi_\Omega(m, n) \right]. \tag{B10}
\]
When \( m \) and \( n \) are both either odd or even, this yields
\[
\Psi(m, n, \Delta z) = -\frac{2}{4D} \left[ 2D \delta_{m,n} \cos(-\gamma m \Delta z) \right], \tag{B11}
\]
otherwise,
\[
\Psi(m, n, \Delta z) = -\frac{2}{4D} \left[ \frac{-2D}{\pi} (\delta_{m,n+1} - \delta_{m,n-1}) \sin(-\gamma m \Delta z) \right]. \tag{B12}
\]
From (B11) and (B12) it is obvious that the mode orthogonality has been lost. It can be partially recovered by multiplying (B10) by \( 2 \cos(\gamma m \Delta z) \), resulting for \( m \) and \( n \) with the same parity
\[
\Psi(m, n, \Delta z) 2 \cos(\gamma m \Delta z) = -\frac{2}{4D} \left[ 2D \delta_{m,n} (1 + \cos(-2\gamma m \Delta z)) \right], \tag{B13}
\]
and for \( m \) and \( n \) with different parity
\[
\Psi(m, n, \Delta z) 2 \cos(\gamma m \Delta z) = -\frac{2}{4D} \left[ \frac{-2D}{\pi} (\delta_{m,n+1} - \delta_{m,n-1}) \sin(-2\gamma m \Delta z) \right]. \tag{B14}
\]
Comparing (B11) with (B13) it is clear that there is a gain in amplitude for \( m = n \), and comparing (B12) with (B14) the amplitude does not change, thus enabling the partial recovery of the modes orthogonality
\[
\Psi(m, n, \Delta z) \cos(\gamma m \Delta z) \approx \Psi(m, n) \frac{1 + \cos(-2\gamma m \Delta z)}{2}. \tag{B15}
\]
A similar result can be obtained if \( \exp(\pm j\gamma m \Delta z) \) is used instead of \( 2 \cos(\gamma m \Delta z) \). In that case a linear phase with \( \Delta z \) will appear for \( m = n \), and (B15) can be generalized to
\[
\Psi(m, n, \Delta z) e^{\pm j\gamma m \Delta z} \approx \Psi(m, n) \frac{W(m, \Delta z)}{2}, \tag{B16}
\]
where $|W(m, \Delta z)|$ is equal to 2 when $\Delta z = 0$ and oscillates around 1 when $\Delta z \neq 0$. 
References


