HIGH-RESOLUTION CLOSED-FORM ESTIMATION OF NORMAL MODE PARAMETERS OF A PARTIALLY SAMPLED WATER COLUMN

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ABSTRACT

The normal mode model is assumed to give a fair representation of sound propagation in shallow water. A simulated experiment is conducted that involves a monochromatic source and a vertical a linear array, with the objective of estimating the parameters of this normal mode model. For this purpose, we have the source transmitting at every sensing depth. Collected measurements are stacked in a first data matrix. The range between source and array is modified, the experiment is repeated and a second data matrix is collected. A special combination of the two data matrices shows an interesting eigen structure, where (non-orthogonal) eigenvectors and eigenvalues turn to be the sought-after sampled model functions and wavenumbers. The so-defined subspace algorithm is not based on (orthogonal) singular vectors and, so, does not require full coverage of the water column, unlike existing subspace algorithms. It also compares advantageously to transform domain techniques which, while not requiring full coverage of the water column, involve impulsive sources, among other limitations.

***Index Terms*—** Shallow water, normal modes, subspace algorithms

I. INTRODUCTION

The normal mode model is commonly used in order to describe propagation of acoustic signals in shallow water. Reliable and fast estimation of the model parameters is of the utmost importance to many underwater observation systems [1], [2], [3], [4]. Existing techniques [5], [6] collect pressure data using vertical linear arrays (VLA) partially covering the water column, and so in order to estimate model functions at the sensor depths, as well as the attached wavenumbers.

Subspace algorithms are popular parameter estimation techniques, especially in antenna array signal processing. They are appreciated mainly for their high resolution, i.e. the ability to provide accurate estimates regardless of the (limited) array size, and so contrarily to transform-domain techniques [2], [3], [5]. Application of subspace algorithms to the shallow water environment has been, so far, less successful because underwater propagation of acoustic signals is far more sophisticated than free-space wireless propagation. For instance, the array output data matrix does not have a special structure unless the array covers the whole water column. Only then, would modal functions appear in the form of orthogonal columns amenable to estimation by matrix singular decomposition [1], [4], [7], [8]. Full coverage of the water column is not possible, not only for practical reasons, but also for physical reasons, in scenarios where waves penetrate the sea bed [3], [5], [9]. We are inspired by non-trivial subspace algorithms (especially those dealing with blind channel estimation) that process (decompose) a matrix that is derived from the array output data matrix, and that has a richer eigen structure.

The proposed algorithm will detect modal functions as eigenvectors, contrarily to existing subspace algorithms [1], [4], [7], [8] where they are detected as singular vectors. In contrast with singular vectors, eigenvectors do not have to be orthogonal. Consequently, the proposed algorithm is free from the constraint of fully sensing the water column. The only existing techniques that accommodate partial sampling of the water column [3], [5] require impulsive sources, large source-array separation (and hence low SNR), user intervention, heavy computation and, more severely, their resolution depend on the array size. Also, the developed method does not require towing the source [1], [9] nor the array [8], [10] which, otherwise, would subject the data to Doppler frequency shift [1], [11], and, potentially, violate the range-independent assumption.

The proposed algorithm will use the data collected during the following steps: 1) the VLA is deployed to sense pressure field at depths of interest, 2) a monochromatic source is activated, successively, at each one of the sensing depths, 3) the collected data is stacked into one first data matrix, 4) the experiment is repeated with a different horizontal spacing between VLA and source, and the new collected data is stacked into a second data matrix. We show how a non-trivial combination of the two matrices exhibits an interesting structure where the sampled model functions appear as eigenvectors, and wavenumbers appear as eigenvalues.

Not only does the resulting algorithm operate under realistic conditions, it also computes the model parameters in a closed-form, search-free and fully-automatic manner, that contrasts with many of the existing heuristic techniques. It, also, does not make any assumption about the statistical distribution of observation noise. The computation burden is that of one eigenvector decomposition and one singular vector decomposition, where the matrix size is determined by the number of array sensing elements.

II. SIGNAL MODEL AND DATA MATRIX

A point in the waveguide is characterized by its coordinates $(x^{(1)}, x^{(2)}, z)$, where $(x^{(1)}, x^{(2)})$ designate the sea surface, z refers to the point's depth [12], [13]. The point position is alternatively determined by (r, ψ, z), where ψ is the angle counter-clockwise from $[O, x^{(1)})$ and r is the horizontal spacing between this point and the reference water column $x^{(1)} = x^{(2)} = 0$, where a mono-chromatic acoustic source is activated at some depth $z = z^S$.

The Q sensors of a VLA, nor densely nor uniformly placed, are maintained at fixed positions $(r, \psi, z) = (r, \theta, z_q)$, $q = 1, \cdots, Q$, where sensor q collects [12]

$$y(z^S, z_q, r) = p(z^S, z_q, r) + \epsilon(z^S, z_q, r)$$

where the randomly distributed noise $\epsilon(z^S, z_q, r)$ affects the noise free observation

$$p(z^S, z_q, r) \hat{=} b_s e^{j\frac{\pi}{4}} \sum_{m=1}^{M} \phi_m(z_q)\phi_m(z^S) \frac{e^{-j\kappa_m r}}{\sqrt{\kappa_m r}} \quad (1)$$

where b_s is an unknown complex amplitude and parameters $\kappa_m, \phi_m(z)$ are relative to the m-th mode. Using matrix notations, $\mathbf{A}(r) \hat{=} \mathrm{Diag}\left[e^{-j\kappa_1 r}/\sqrt{\kappa_1 r}, \cdots, e^{-j\kappa_M r}/\sqrt{\kappa_M r}\right]$, so that we have

$$p(z^S, z_q, r) = b_s e^{j\frac{\pi}{4}} [\phi_1(z_q), \cdots, \phi_M(z_q)] \mathbf{A}(r) \begin{bmatrix} \phi_1(z^S) \\ \vdots \\ \phi_M(z^S) \end{bmatrix} \quad (2)$$

We stack measurements from the VLA into vector

$$\hat{=} b_s e^{j\frac{\pi}{4}} \Phi \mathbf{A}(r) \begin{bmatrix} \phi_1(z^S) \\ \vdots \\ \phi_M(z^S) \end{bmatrix} + \begin{bmatrix} \epsilon(z^S, z_1, r) \\ \vdots \\ \epsilon(z^S, z_Q, r) \end{bmatrix}$$

where columns of $\Phi \hat{=} \begin{bmatrix} \phi_1(z_1) & \cdots & \phi_M(z_1) \\ \vdots & & \\ \phi_1(z_Q) & \cdots & \phi_M(z_Q) \end{bmatrix}$ are the modal functions sampled at $z_1, \cdots, z_Q$. Now, let's imagine that we set source depth as $z^S = z_1$, and collect $y(z_1, z_1, r), \cdots, y(z_1, z_Q, r)$. Then, we move the source to depth $z^S = z_2$ and collect $y(z_2, z_1, r), \cdots, y(z_2, z_Q, r)$, hence collecting Q^2 measurements arranged into data matrix

$$\mathbf{Y}(r) \hat{=} \begin{bmatrix} y(z_1, z_1, r) & \cdots & y(z_Q, z_1, r) \\ \vdots & & \\ y(z_1, z_Q, r) & \cdots & y(z_Q, z_Q, r) \end{bmatrix} \quad (3)$$

$$\begin{aligned} &= b_s e^{j\frac{\pi}{4}} \Phi \mathbf{A}(r) \Phi^T + \begin{bmatrix} \epsilon(z_1, z_1, r) & \cdots & \epsilon(z_Q, z_1, r) \\ \vdots & & \\ \epsilon(z_1, z_Q, r) & \cdots & \epsilon(z_Q, z_Q, r) \end{bmatrix} \\ &\hat{=} \mathbf{P}(r) + \mathbf{E}(r) \end{aligned} \quad (4)$$

Non-Hermitian $\mathbf{P}(r)$ is symmetric [but not $\mathbf{E}(r)$], a property that we will use in the next section to reduce the impact of noise.

Finally, we reasonably assume columns of Φ are linearly independent, but not necessarily orthogonal (which, otherwise, requires sensing the total water column). It follows that $\mathbf{P}(r)$ is rank-deficient if $Q \geq M$, which is achievable at least for low frequency sources [8]. At last, we denote $\Phi^\natural$ as the real-valued $M \times Q$ Moore pseudo-inverse of Φ verifying $\Phi^\natural \Phi = \mathbf{I}$. Advantageously, we do not make any statistical (normality and/or uniformity) assumption about noise, which is safe enough since noise sources are more frequently impulsive [14].

III. ALGORITHM DERIVATION

The proposed method is based on the execution of the above measurement procedure at two different locations with different horizontal spacing betwen source and VLA. The source is activated at consecutive depths $z_1, \cdots, z_Q$. VLA-to-source horizontal spacing is R_1 in the 1st experiment and R_2 in the 2nd experiment. We obtain two data matrices, as in (3), noted $\mathbf{Y}_k \hat{=} \mathbf{Y}(R_k)$, $k = 1, 2$. Each verifies $\mathbf{Y}_k = b_s e^{j\frac{\pi}{4}} \Phi \mathbf{A}(R_k) \Phi^T$ and has the pseudo-inverse $\mathbf{Y}_k^\natural = b_s^{-1} e^{-j\frac{\pi}{4}} \Phi^{\natural, T} \mathbf{A}^{-1}(R_k) \Phi^\natural$. We realize that

$$\mathbf{Y}_k \mathbf{Y}_l^\natural = \sqrt{\frac{R_l}{R_k}} \Phi \mathrm{Diag}\left(e^{j\kappa_1(R_l - R_k)}, \cdots, e^{j\kappa_M(R_l - R_k)}\right) \Phi^\natural$$

is similar to $\sqrt{R_l/R_k}\mathrm{Diag}\left(e^{j\kappa_1(R_l - R_k)}, \cdots, e^{j\kappa_M(R_l - R_k)}, 0, 0, \cdots\right)$. Its non-zero eigen values $\sqrt{R_l/R_k} e^{j\kappa_m(R_l - R_k)}$, $m = 1, \cdots, M$ are associated to eigenvectors that are nothing else but the columns of Φ. Notice that because $\mathbf{Y}_k \mathbf{Y}_l^\natural$ is not Hermitian, its eigen vectors should not be orthogonal.

The above rationale guarantees exact calculation of normal mode parameters from noise-free measurements. The presence of noise calls for the following adaptations. First, if $R_1 < R_2$, than $\mathbf{Y}_1$ is expected to be less affected by noise and so, will be the one we invert, to finally process

$$\mathbf{Y}_2 \mathbf{Y}_1^\natural = \sqrt{\frac{R_1}{R_2}} \Phi \mathrm{Diag}\left(e^{j\kappa_1(R_1 - R_2)}, \cdots, e^{j\kappa_M(R_1 - R_2)}\right) \Phi^\natural$$

Second, symmetric $\mathbf{P}_1 \hat{=} \mathbf{P}(R_1)$ and $\mathbf{P}_2 \hat{=} \mathbf{P}(R_2)$ allow us to rewrite $\mathbf{Y}_k$ as $(\mathbf{Y}_k + \mathbf{Y}_k^T)/2$, equal to $\mathbf{Y}_k = \mathbf{P}_k + [\mathbf{E}(R_k) + \mathbf{E}^T(R_k)]/2$, where the noise power is halved. Finally, the estimation algorithm is executed as follows:

1) Collect $Q \times Q$ matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$.
2) Update $\mathbf{Y}_k$ as $(\mathbf{Y}_k + \mathbf{Y}_k^T)/2$ for $k = 1, 2$.
3) For $m = 1, \cdots, M$, let σ_m be the largest singular value of $\mathbf{Y}_1$. Let $\mathbf{u}_m$ and $\mathbf{v}_m$ be, respectively, the associated left and right unit-norm singular vectors.
4) Calculate $\mathbf{Y}_1^\natural$ as $\sum_{m=1}^{M} \sigma_m^{-1} \mathbf{v}_m \mathbf{u}_m^H$.
5) For $m = 1, \cdots, M$, let λ_m be the eigen value of $\mathbf{Y}_2 \mathbf{Y}_1^\natural$, with the m-th largest magnitude, and associated to the unit-norm eigen vector $\mathbf{w}_m$.

6) Estimate $(R_1 - R_2)\kappa_m$ as $\arg(\lambda_m)$, selected in $[0, 2\pi]$.
7) Estimate $[\phi_m(z_1), \cdots, \phi_m(z_P)]^T$ by $\mathbf{w}_m$.

It is clear that the obtained estimates of the modal functions and the wavenumbers do suffer from some indeterminacy, which is, by the way, common to both subspace [8] and transform-domain [3] techniques. While countermeasures exist [8], [9], we do not use any here. We will evaluate estimation performance independently from the ambiguity removal technique which will greatly depend on the application.

IV. NUMERICAL SIMULATIONS

The KRAKEN normal mode propagation model is used in a 100 [m] depth waveguide, excited by a mono-chromatic source emitting a tone at various frequencies: $50, 100$ and 150 [Hz]. The acoustic pressure field was generated for a depth-range grid of $[10, 90] - [0, 1000]$ [m] with 51×101 equally spaced receivers. Given that the source is activated at consecutive depths $z_1, \cdots, z_Q$, we have a total of Q^2 measurements $p(z_q, z_{q'}, r)$, for $z_q, z_{q'} = z_1, \cdots, z_Q$, grouped into a $Q \times Q$ matrix $\mathbf{P}(r)$. The average signal power is defined as $(1/Q^2) \sum_{q,q'=1}^{Q} |p(z_q, z_{q'}, r)|^2$.

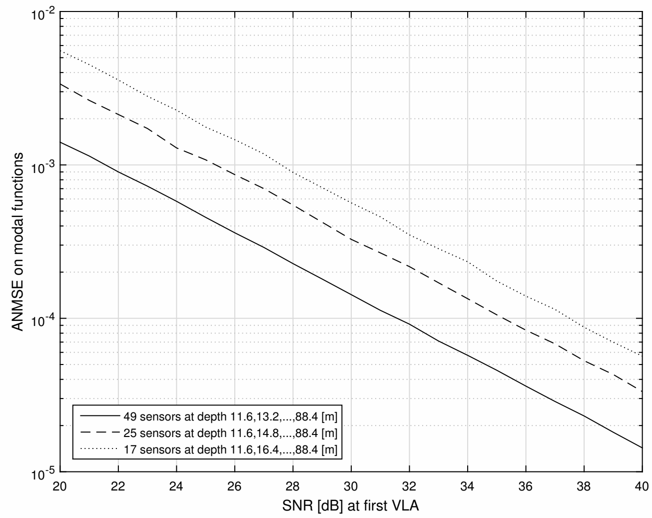
We corrupt the received data field with zero-mean complex-valued circular Additive White Gaussian Noise (AWGN), following (4). We assume noise components to be independent at every sensor, with uniform power $\mathrm{E}\left[|\epsilon(z_p^R, z^S, r)|^2\right] = \sigma_n^2$. Subsequently, for a VLA deployed at range r, the reported SNR is defined as $[1/(Q^2\sigma_n^2)] \sum_{q,q'=1}^{Q} |p(z_q, z_{q'}, r)|^2$ [12], [13, Sec. VI]. We decided to place the two VLAs at 200 [m] and 400 [m] from the source.

The presence of noise will result in an estimate $\mathbf{w}_m$ that is roughly co-linear to the exact m-th modal function $[\phi_m(z_1), \cdots, \phi_m(z_P)]^T$. It is custom, when a vector $\mathbf{x}$ is estimated by $\hat{\mathbf{x}}$ up to an unknown multiplicative constant, to use the following normalized Mean Square Error (MSE) defined as $\left(1/\|\mathbf{x}\|^2\right) \min_\beta \|\mathbf{x} - \beta\hat{\mathbf{x}}\|^2$, proved in [15] to be equal to $1 - \left[\left|\mathbf{x}^H\hat{\mathbf{x}}\right| / (\|\mathbf{x}\|\|\hat{\mathbf{x}}\|)\right]^2$. It is normal in the sense that it ranges between 0 ($\mathbf{x}$ and $\hat{\mathbf{x}}$ are co-linear) and 1 ($\mathbf{x}$ and $\hat{\mathbf{x}}$ are orthogonal). Applied to our estimates of the M modal functions, a global normalized performance measure is calculated as

$$1 - \frac{1}{M} \sum_{m=1}^{M} \left(\frac{\left|[\phi_m(z_1), \cdots, \phi_m(z_Q)]\, \mathbf{w}_m^R\right|}{\|\,[\phi_m(z_1), \cdots, \phi_m(z_Q)]\,\|\|\mathbf{w}_m^R\|} \right)^2$$

This normalized MSE on modal functions is averaged over 1000 Monte Carlo runs, and reported as the Averaged Normalized MSE (ANMSE).

As described earlier, the algorithm delivers $\arg(\lambda_m)$ as estimates of $(R_1 - R_2)\kappa_m, m = 1, \cdots, M$, modulo 2π, which does not uniquely characterize the wavenumbers. In order to assess performance in a way that does not depend on how we solve this ambiguity problem, we decided to report in the different figures $\left[(R_1 - R_2)\kappa_m - \arg(\lambda_m)\right]^2 / (4\pi^2)$ as a performance indicator that we, abusively, term as ANMSE on wavenumbers.



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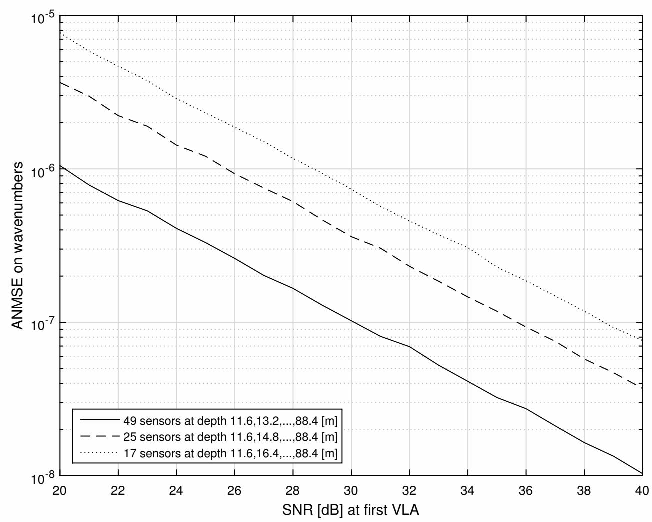
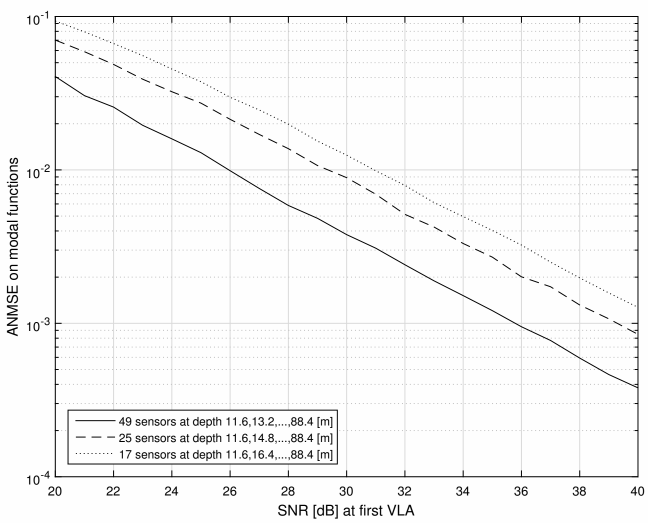


(b)

Fig. 1. Average normalized mean square error vs. SNR in dB for the Pekeris waveguide at $f = 50$ [Hz] for 17, 25 and 49 hydrophones for modal functions (a) and modal wavenumbers (b).

In order to test the algorithm capabilities, we first used a Pekeris waveguide with a water column sound speed of 1500 [m/s] and a bottom half-space with a compressional speed of 1800 [m/s] and a density $\rho = 1.8$ [Kg/m^3]. The ANMSE results are shown in figures 1 to 3 as a function of SNR ranging from 20 to 40 [dB] for variable 17, 25 and 49



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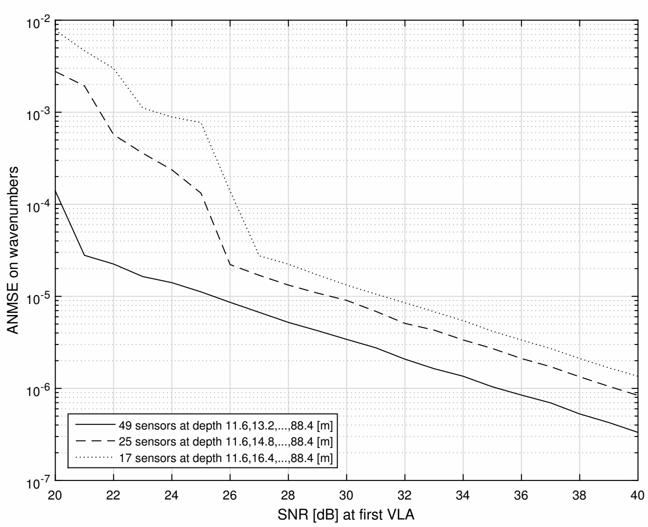
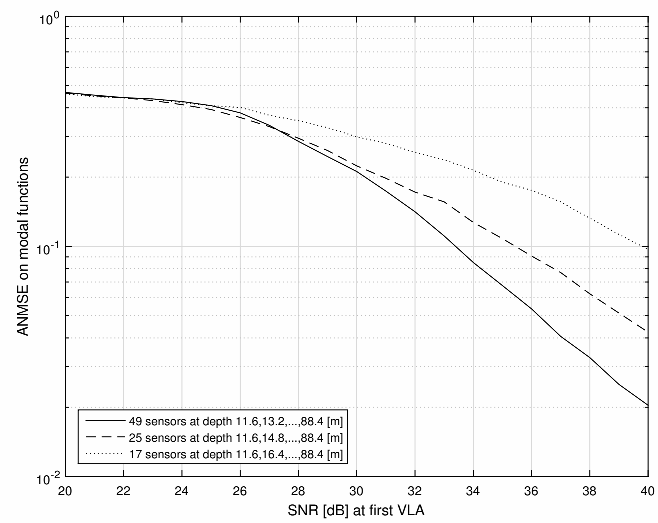


(b)

Fig. 2. Average normalized mean square error vs. SNR in dB for the Pekeris waveguide at $f = 100$ [Hz] for 17, 25 and 49 hydrophones for modal functions (a) and modal wavenumbers (b).



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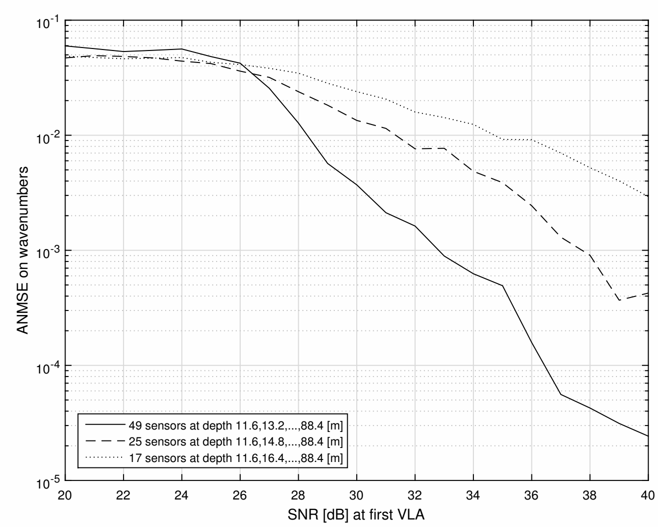


(b)

Fig. 3. Average normalized mean square error vs. SNR in dB for the Pekeris waveguide at $f = 150$ [Hz] for 17, 25 and 49 hydrophones for modal functions (a) and modal wavenumbers (b).

hydrophone VLA's. There, the estimation performance, for modal functions (a) and modal wavenumbers (b), is perfect at 50 and 100 Hz, with the waveguide supporting 4 and 9 modes, respectively, and degrades slightly at 150 Hz, when the number of modes reaches 13.

One of the pending questions is that of *leaking* waveguides where a significant part of the energy is lost into the sediment and does not reach the receiver, leading to non-orthogonal normal mode functions over the span of the VLA's. In order to test the performance of the algorithm, we simulated a downward refracting profile with a soft 20 [m] thick sediment and a non-homogeneous sound speed profile with a 30 [m] depth mixed layer, as shown in the diagram of figure 4. This scenario supports 5, 11 and 16 modes at 50, 100 and 150 [Hz], respectively. ANMSE performance results are shown in figure 6 for the modal functions (a) and the modal wavenumbers (b). It can be noticed that the performance is degraded when compared with that of the

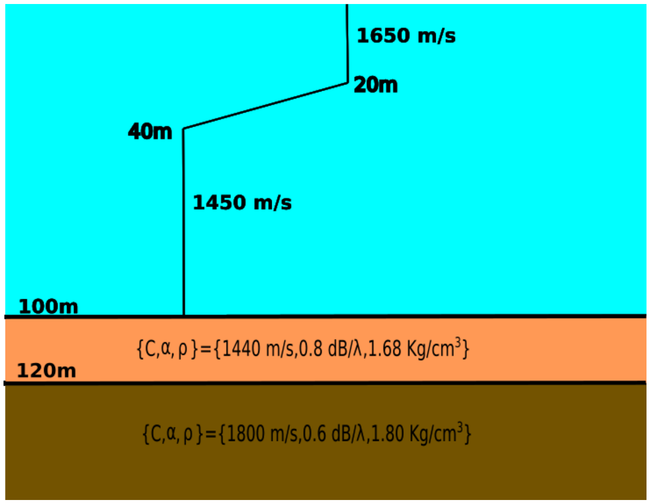


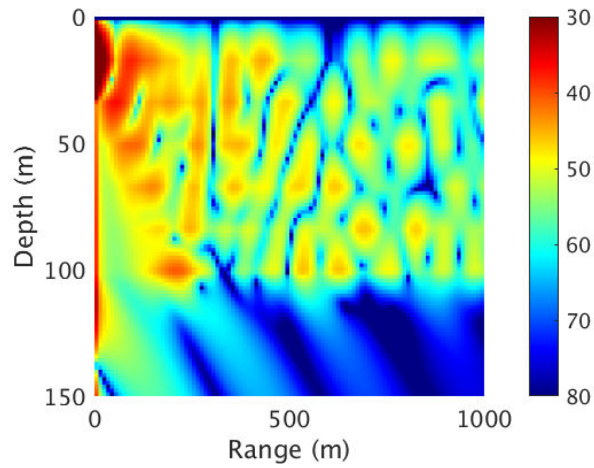
Fig. 4. Scenario for the soft bottom case with a low compressional velocity 20 [m] thick sediment.

Pekeris case and is only reasonable for SNR $>$ 30 [dB], or so. Bottom absorption and signal has therefore a clear effect on performance.

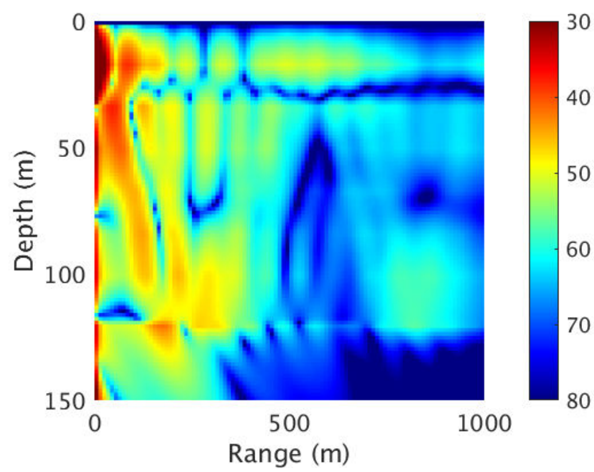
In figures 1, 2, 3 and 4, we illustrated the effect of array sparsity on the estimation accuracy. In figure 7, we study the estimation accuracy when data is collected using a dense VLA, and so for all possible positions and lengths of the dense VLA. There, a performance measure in [dB] is calculated as $-10\log_{10}(ANMSE)$, for both modal functions and wavenumbers, and shows, as expected, that the longer the array, the better is the algorithm performance.

V. CONCLUSION

In a shallow water, where acoustic propagation is modeled using normal modes, we deploy a vertical linear array and activate a distant mono-chromatic source. The collected data is fed to a subspace algorithm to infer about the parameters of the normal mode model, for instance, the modal functions (evaluated at the sensing depths) and the associated wavenumbers. While existing techniques tow the array over a certain distance and process the raw array output matrix, we conduct an original experiment: We activate the source at each of the sensing depth, we repeat the manipulation at another location. A non-trivial combination of the two so-collected data matrices exhibits an interesting eigenstructure that wouldn't appear for the raw data matrix unless the full water column is sensed. It follows that the proposed algorithm accumulates many attractive features: it is computationally efficient (no iterations, nor systematic search), robust (with no assumption other than using more sensors than modes), and high-resolution (i.e., its accuracy is limited only by noise, not by the array size). Finally, this proposal proves that algebraic methods can be as powerful for acoustic systems, as they are for radio-frequency systems.



(a)



(b)

Fig. 5. Transmission loss of the *leaking* waveguide (b) compared to the Pekeris waveguide at 150 [Hz].

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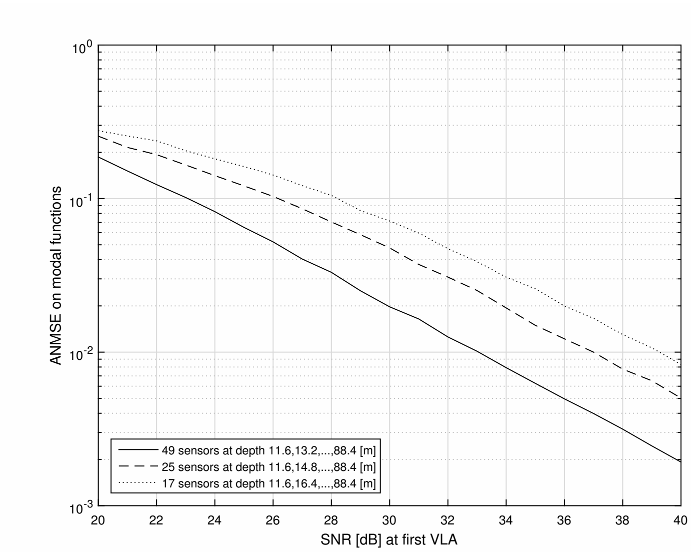

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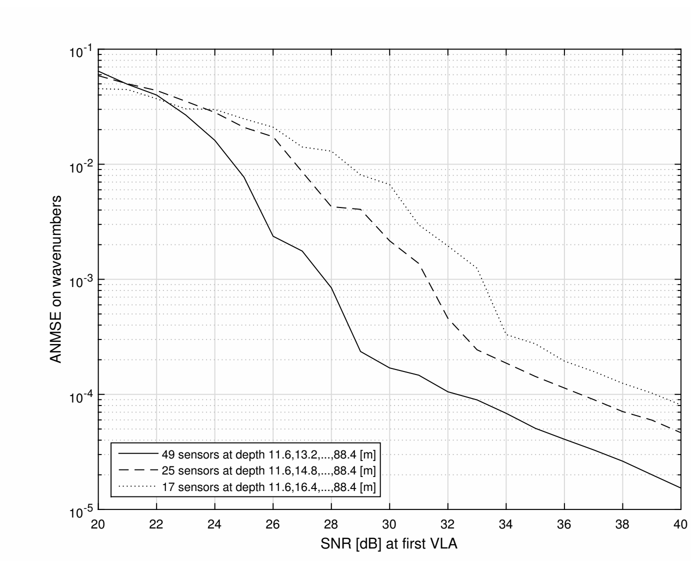

(b)

Fig. 6. Average normalized mean square error vs. SNR in dB for the soft bottom waveguide at $f = 50$ [Hz] for 17, 25 and 49 hydrophones for modal functions (a) and modal wavenumbers (b).

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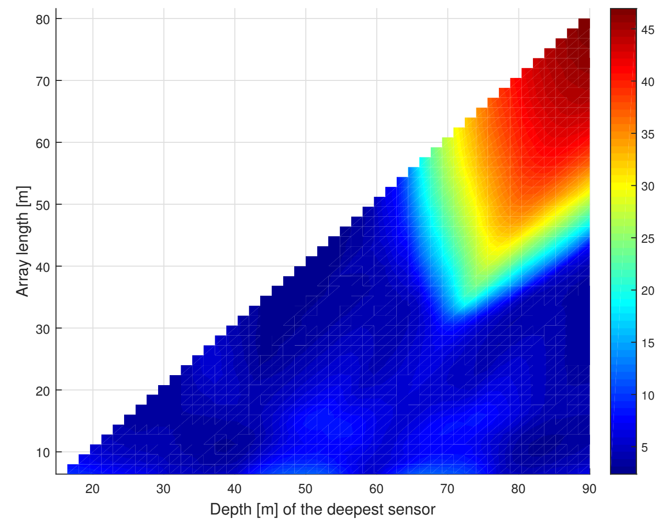

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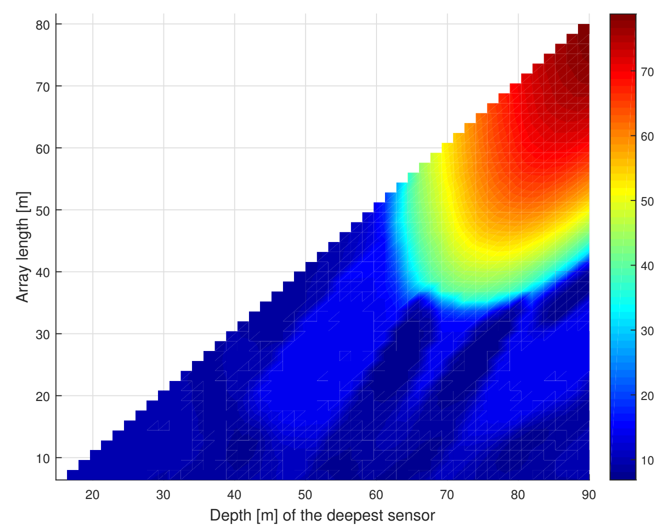

(b)

Fig. 7. Average normalized mean square error for the Pekeris waveguide at $f = 50$ [Hz] for modal functions (a) and modal wavenumbers (b), for all lengths and positions of a dense VLA, in the presence of AWGN noise characterized by a standard deviation $\sigma_n = 10^{-4}$.

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