Closed-Form Estimation of Normal Modes From a Partially Sampled Water Column

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***Abstract*—The output of a vertical linear array is used to infer about the parameters of the normal mode model that describes acoustic propagation in a shallow water. Existing subspace algorithms perform singular vector decomposition of the array data matrix to estimate the sampled model functions. Estimates are exact only if the sensing array is totally covering the water column. We design a new subspace algorithm free from this very restrictive requirement. We use two short hydrophone arrays and activate a monochromatic source at different depths. Estimates of both the modal functions and the wave numbers are obtained in a fully automatic and search-free manner. The algorithm can be qualified as truly high resolution in the sense that, while using short sensing arrays, estimation error becomes arbitrarily low if observation noise is arbitrarily low. This method compares advantageously to existing subspace techniques, as well as transform-domain techniques that require impulsive sources, among other constraints. With two (eigen and singular) vector decompositions, the proposed technique has the complexity of a regular subspace algorithm.**

***Index Terms*—Normal modes, shallow water, subspace algorithms, vertical linear arrays.**

I. Introduction

SHALLOW water is a challenging propagation environment that requires the design of dedicated signal processing techniques to perform efficiently the different tasks of an underwater observation system: identification, localization, communication, or also inversion. Such techniques will not be efficient until they are based on a suitable propagation model and that the model parameters are accurately estimated. In this context, the normal mode model emerges as a popular mathematical description of acoustic propagation in an underwater waveguide. The different constituent modes are described, each, by a modal depth function and a wave number, reflecting the complexity of the model and the challenges posed to the estimation algorithm.

Accurate estimation of the model parameters is very critical to the performance of geoacoustic inversion [1]–[3], and to source localization algorithms [4], [5], especially high resolution ones. Current techniques identify sampled modal functions (evaluated at the sensor depths) using the output of an array of hydrophones arranged in vertical linear array (VLA) [1], [6]–[9] or in horizontal linear array (HLA) [3], [5], [10] configurations. On one hand, HLA-based techniques are very restrictive about the source location. It has to be maintained at end-fire, strictly [5], [11] or loosely [10]. Also restrictive is the fact that, the HLA needs to be deployed close to the node of any active modal function [10, Sec. IV].

VLA-based techniques come with some shortcomings to which we will try to remedy. We start by discarding the least attractive techniques which include heuristic techniques [12], [13], or a technique that requires the processing of coherent signals over long periods of time [11]. Most VLA-based techniques can be classified as transform-domain techniques or subspace techniques. Transform-domain techniques [2], [3], [9] are search-based and their resolution is limited by the array size. Subspace techniques operate on (averaged) data arranged in rank deficient matrices. The associated (noise) subspace is used to build identification MUSIC-like cost functions. Cost functions are optimized by exhaustive-search [5], unless a uniform VLA is used whose Hankel-structured data matrix allows for closed-form estimation [1].

The major limitation of existing subspace techniques is the need to deploy a dense VLA spanning the whole water column, and so to collect orthogonal modal functions [1], [7], [8]. Subsequently, estimation fails if the array is sparse or short, as discussed in [6, App. A], but also if some modes propagate inside the seabed where they cannot be sensed. This has long been thought of as an inherent drawback of subspace techniques [2], [9]. In this paper, we show that this intuition is not true. We design a new subspace technique that does not require orthogonality of modal functions, and hence no full sampling of the water column. Furthermore, this technique is closed-form, not requiring a systematic multidimensional search. This is made possible thanks to the use of eigen vector decomposition (EVD) instead of singular value decomposition (SVD). Singular vectors are always orthogonal, contrarily to eigenvectors that do not, in general, have to verify orthogonality. Our algorithm determines the normal mode parameters (sampled modal functions and wave numbers), like [9] and [13], but unlike [1], [5], [10], and [11] that estimate wave numbers only; or [6] and [7] that estimate modal functions only.

To our best knowledge, the only existing techniques that extract normal modes and, at the same time, accommodate partial sampling of the water column are those presented in [2] and [9]. They operate in the transform domain and require impulsive sources (less convenient than continuous wave and triangular pulses [14]) that has to traverse a significant range

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interval [9], or to be activated very far from the sensors [2]. These techniques are based on modal separability that is more pronounced at large ranges where, unfortunately, signal-to-noise ratio (SNR) is rather low, and requires the use of nonlinear processing techniques (warping [2] or masking [9]) that operate under restrictive conditions [2, eqs. (7) and (8)] [9, eqs. (23)–(25)]. Also, they are not fully automatic as they involve some user-defined parameters. Computation requirements of [2], [3], and [9] are also high as Fourier transforms or SVDs are repeatedly computed. These drawbacks affect the practicality of these mode extraction methods as well as their resolution, resulting in a limited ability to extract weakly excited modes [9].

We start by introducing, in Section II, the observation model and highlight the structure of the data matrix collected by means of the VLA, expressed in terms of modal functions and wave numbers. In Section III, we present the theoretical development of the proposed algorithm, first in the noise-free case where the algorithm delivers exact estimates, then in the practical noise case that calls for some adaptations. A series of simulations are reported in Section IV to demonstrate the efficiency of the proposed method under different conditions of SNR, range, and depth. Finally, a conclusion is given in Section V to summarize the main features of the developed algorithm.

Throughout this paper, we adopt the following notations. Matrices (resp. vectors) are represented by bold upper (resp. lower) case characters. Vectors are, by default, in column orientation and their Euclidean norm is referred by $\|\cdot\|$. We use T, H, $*$, -1, and $\natural$ to denote matrix transpose, transconjugate, conjugate, inverse, and Moore pseudoinverse, respectively. Entry at row i and column j of matrix $\mathbf{A}$ is denoted as $[\mathbf{A}]_{ij}$. $\mathrm{Diag}\,(d_1, d_2, \ldots)$ denotes the diagonal matrix with $d_1, d_2, \ldots$ along the diagonal. $\mathbf{0}_{a,b}$ is the $a \times b$ zero matrix. $\mathbf{I}_a$ is the $a \times a$ identity matrix. Dimensions are dropped when they can be inferred from the context.

II. Data Model

We adopt a coordinate system [15], [16] $[O, x^{(1)}, x^{(2)}, z)$ where $[O, x^{(1)}, x^{(2)})$ designates the sea surface. A point in the waveguide is characterized by its coordinates $(x^{(1)}, x^{(2)}, z)$ where z refers to the point's depth, or equivalently by (r, ψ, z), where ψ is the angle counter-clockwise from $[O, x^{(1)})$ and r is the horizontal spacing between this point and the reference water column $x^{(1)} = x^{(2)} = 0$. There, an acoustic narrowband source located at $(x^{(1)}, x^{(2)}, z) = (0, 0, z^S)$ is activated and the environing pressure field is measured by means of Q sensors disposed as a VLA, not in the immediate vicinity of the source.

Sensors are maintained at positions $(r, \psi, z) = (r, \theta, z_q)$, $q = 1, \ldots, Q$, so that sensor q collects [15, eq. (3)]

$$y(z^S, z_q, r) = b_s e^{j\frac{\pi}{4}} \sum_{m=1}^{M} \phi_m(z_q)\phi_m(z^S)\frac{e^{-j\kappa_m r}}{\sqrt{\kappa_m r}} + \epsilon(z^S, z_q, r) \tag{1}$$

where b_s is an unknown complex amplitude, while model parameters κ_m and $\phi_m(z)$ correspond to the mth wave number and the mth modal function, respectively. The randomly distributed $\epsilon(z^S, z_q, r)$ refers to the noise collected at depth z_q and range r when the source is activated at depth z^S. We focus on the noise-free pressure readings

$$p\left(z^S, z_q, r\right) \hat{=} b_s e^{j\frac{\pi}{4}} \sum_{m=1}^{M} \phi_m(z_q)\phi_m(z^S)\frac{e^{-j\kappa_m r}}{\sqrt{\kappa_m r}}.$$

This is to be rewritten using matrix notations, more convenient for the development of a subspace algorithm. For instance, if $\mathbf{A}(r) \hat{=} \mathrm{Diag}\left[e^{-j\kappa_1 r}/\sqrt{\kappa_1 r}, \ldots, e^{-j\kappa_M r}/\sqrt{\kappa_M r}\right]$, then we have

$$p\left(z^S, z_q, r\right) = b_s e^{j\frac{\pi}{4}} [\phi_1(z_q), \ldots, \phi_M(z_q)] \times \mathbf{A}(r)\left[\phi_1\left(z^S\right), \ldots, \phi_M\left(z^S\right)\right]^T \tag{2}$$

referring to the pressure induced at sensor q by the source located at depth z^S, separated by horizontal distance r. Our objective is to determine modal functions $\phi_1(z), \ldots, \phi_M(z)$ and wave numbers $\kappa_1, \ldots, \kappa_M$, using a technique that is exact (if there were no noise) and blind (i.e., based solely on a finite number of field measurements).

First, let us stack all measurements collected by the VLA, while the source is maintained at depth z^S, into the vector

$$\begin{aligned}
\begin{bmatrix} y(z^S, z_1, r) \\ \vdots \\ y(z^S, z_Q, r) \end{bmatrix} &= b_s e^{j\frac{\pi}{4}} \begin{bmatrix} \phi_1(z_1) & \cdots & \phi_M(z_1) \\ \vdots & \vdots & \\ \phi_1(z_Q) & \cdots & \phi_M(z_Q) \end{bmatrix} \times \mathbf{A}(r) \begin{bmatrix} \phi_1(z^S) \\ \vdots \\ \phi_M(z^S) \end{bmatrix} + \begin{bmatrix} \epsilon(z^S, z_1, r) \\ \vdots \\ \epsilon(z^S, z_Q, r) \end{bmatrix} \\
&\hat{=} b_s e^{j\frac{\pi}{4}} \Phi(z_1, \ldots, z_Q)\mathbf{A}(r) \begin{bmatrix} \phi_1(z^S) \\ \vdots \\ \phi_M(z^S) \end{bmatrix} + \begin{bmatrix} \epsilon(z^S, z_1, r) \\ \vdots \\ \epsilon(z^S, z_Q, r) \end{bmatrix}
\end{aligned}$$

where column m of the $Q \times M$ matrix $\Phi(z_1, \ldots, z_Q)$ represents the mth modal function sampled at $z_1, \ldots, z_Q$. Now, let us imagine that we set source depth as $z^S = z_1$, and collect $y(z_1, z_1, r), \ldots, y(z_1, z_Q, r)$. Then, we move the source to depth $z^S = z_2$ and collect $y(z_2, z_1, r), \ldots, y(z_2, z_Q, r)$. By repeating this procedure at all sensing depths $z_1, \ldots, z_Q$, we collect Q^2 measurements $y(z_q, z_{q'}, r), q, q' = 1, \ldots, Q$, that we arrange into the $Q \times Q$ data matrix

$$\mathbf{Y}(r) \hat{=} \begin{bmatrix} y(z_1, z_1, r) & \cdots & y(z_Q, z_1, r) \\ \vdots & & \vdots \\ y(z_1, z_Q, r) & \cdots & y(z_Q, z_Q, r) \end{bmatrix} \tag{3}$$

$$= b_s e^{j\frac{\pi}{4}} \Phi(z_1, \ldots, z_Q)\mathbf{A}(r)\Phi^T(z_1, \ldots, z_Q)$$

$$+\begin{bmatrix}\epsilon(z_1,z_1,r) & \cdots & \epsilon(z_Q,z_1,r)\\ \vdots & & \vdots\\ \epsilon(z_1,z_Q,r) & \cdots & \epsilon(z_Q,z_Q,r)\end{bmatrix} \tag{4}$$

$$\hat{=}\,\mathbf{P}(r)+\mathbf{E}(r). \tag{5}$$

Notice that the non-Hermitian matrix $\mathbf{P}(r)$ is symmetric [but not $\mathbf{E}(r)$], which is a property that we will use in Section III-C to reduce the impact of noise.

To develop our subspace estimation algorithm, we make use of the fact, that we assume is verified with probability one, that a tall matrix $\Phi(z_1,\ldots,z_Q)$ is full column rank. On the contrary, we do not assume columns of $\Phi(z_1,\ldots,z_Q)$ to be orthogonal, which relieves from the need for a dense and total coverage of the water column. Thanks to the above condition, the $Q\times Q$ matrix $\mathbf{P}(r)$ has rank M if we have $Q\geq M$, i.e., more sensors than modes. This condition is frequently assumed in subspace methods [5], [7], [10] and easily achievable in practice, at least for low-frequency sources [7]. To simplify notation, from now on, we denote $\Phi\,\hat{=}\,\Phi(z_1,\ldots,z_Q)$, and $\Phi^{\natural}$ as its real-valued $M\times Q$ pseudoinverse. The pair verifies $\Phi^{\natural}\Phi=\mathbf{I}$.

III. Algorithm Development

A. Algorithm Classification

The proposed algorithm can be classified as a deterministic subspace algorithm. Deterministic algorithms are those who exploit structural properties of data matrices [17], as opposed to statistical methods which are based on structural properties of correlation matrices and/or cross-spectral density matrices. The data matrix is obtained by stacking array outputs collected at different ranges [1, eq. (3)], [7, eq. (1)], while the cross-spectral density matrix is obtained as the outer product of the array outputs, averaged over many frequency bins [8], [9], over a range interval [7, eq. (18)], [9], [10, Sec. III], or over time [6].

Whether the processed matrix is a data matrix, a correlation matrix, or a cross-spectral density matrix, the subspace algorithm will require the array to be long enough in order for the processed matrix to be rank-deficient, enabling the distinction of noise and signal subspaces. In addition, it is required that either the source [1], [6] or the receiver (VLA [7] or HLA [10]) moves horizontally over a significant range interval. Horizontal movements cause a Doppler frequency shift (deliberately ignored in [1, Sec. III]) that translates into a bias of the wave number estimates [18], and, more seriously, may violate the range-independent assumption. The proposed technique operates in a less restrictive manner. We merely have the experiment repeated twice, each time with a different range between source and VLA. Since there is no need to move the array over a long range interval as in [7], our design is more robust to currents that may displace the sensors away from their nominal positions [11], [19].

Because they are developed under the noise-free condition, deterministic algorithms, advantageously, do not make any assumption about the statistical properties of noise. This contrasts with (subspace) algorithms and (CRB-based) analyzes that deal with averaged array outputs. As such, they require statistical characterization of the observation noise. For instance, they assume the noise components to satisfy one or many of the following assumptions: noise is normally distributed [15], [16], [20], noise power is the same at all depths [15], [20], noise power is the same at all ranges [5], and noise power is dependent on the sensor depth, but known [16]. Some of these assumptions are unrealistic because noise sources often happen to be impulsive [21], and there is a significant gap between surface noise and bottom noise, especially at lower frequencies.

Among existing deterministic algorithms, the one in [7] requires prior normalization of the pressure data [7, eq. (13)], which is not without impact on noise. It also comes with many restrictive conditions [7, eqs. (16) and (17)] on source-VLA range (but not only). Because of the many approximations, this algorithm does not deliver exact estimates even in the absence of measurement noise, admittedly failing to estimate high-order modes. At last, this method does not deliver estimates of the mode wave numbers. The other deterministic algorithm [1] openly requires dense and full sampling of the water column.

The proposed algorithm will be developed, first, assuming noise-free measurements, a requirement for exact determination of the model parameters. Then, a noise-compatible version of the algorithm is detailed in Section III-C, and tested in Section IV, to study the impact of observation noise on estimation accuracy.

B. Noise-Free Case

Let us assume two experiments are conducted, where every time VLA sensors are maintained at depths $z_1,\ldots,z_Q$. The source is activated at, consecutively, $z_1,\ldots,z_Q$, or, equivalently, a vertical source array [22] is deployed at the same depths $z_1,\ldots,z_Q$, whose elements are activated one at a time. VLA-source horizontal spacing is R_1 in the 1st experiment and R_2 in the 2nd experiment. Data matrices collected from these experiments, arranged as in (3), are labeled $\mathbf{Y}_k\,\hat{=}\,\mathbf{Y}(R_k)$, $k=1,2$. Assuming a range independent environment, w.r.t. (4), the two collected data matrices verify, for $k=1,2$, the same structure $\mathbf{Y}_k=b_se^{j\frac{\pi}{4}}\Phi\mathbf{A}(R_k)\Phi^T$. To each, we attach the pseudoinverse $\mathbf{Y}_k^{\natural}$ that can be proved to be equal to $b_s^{-1}e^{-j\frac{\pi}{4}}\Phi^{\natural,T}\mathbf{A}^{-1}(R_k)\Phi^{\natural}$. Interestingly, the data matrix

$$\mathbf{Y}_k\mathbf{Y}_l^{\natural}=\sqrt{\frac{R_l}{R_k}}\Phi\mathrm{Diag}\left(e^{j\kappa_1(R_l-R_k)},\ldots,e^{j\kappa_M(R_l-R_k)}\right)\Phi^{\natural} \tag{6}$$

is similar to matrix

$$\sqrt{R_l/R_k}\mathrm{Diag}(e^{j\kappa_1(R_l-R_k)},\ldots,e^{j\kappa_M\ (R_l-R_k)},0,0,\ldots).$$

Its nonzero eigenvalues $\sqrt{R_l/R_k}e^{j\kappa_m(R_l-R_k)}, m=1,\ldots,M$ are associated to eigenvectors that are nothing but the columns of Φ (a detailed proof is given in the appendix), i.e., the sought-after modal functions, observed at sensor locations. Such eigenvectors can be obtained from $\mathbf{Y}_k\mathbf{Y}_l^{\natural}$ using an appropriate algorithm [23], hence computing scaled estimates of modal functions.

Notice that because $\mathbf{Y}_k\mathbf{Y}_l^{\natural}$ is not Hermitian, its eigenvectors should not be orthogonal. Also, notice that the algorithm needs to know about the spacing R_1-R_2 between the two VLAs, not the actual range values.

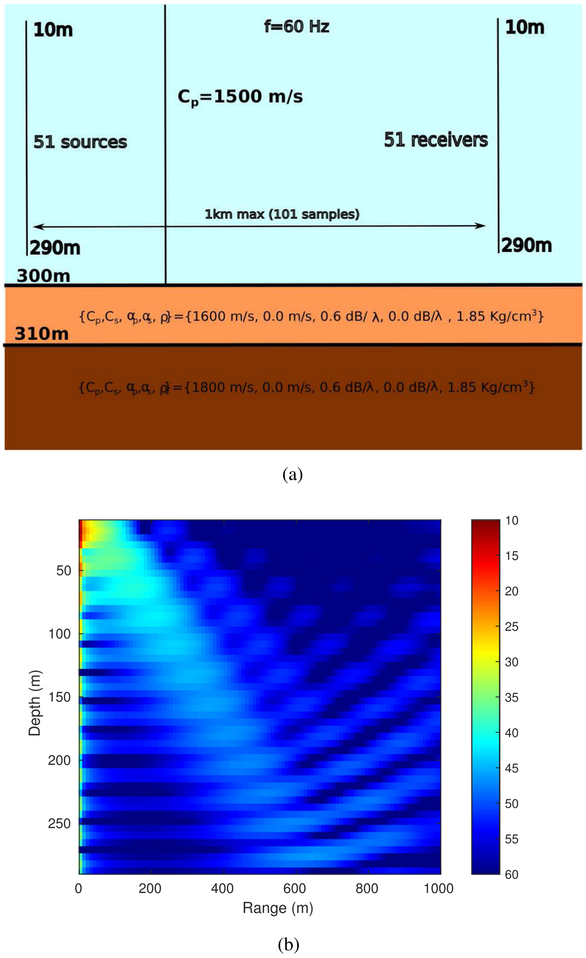


Fig. 1. Experimental settings. (a) Waveguide. (b) Transmission loss.

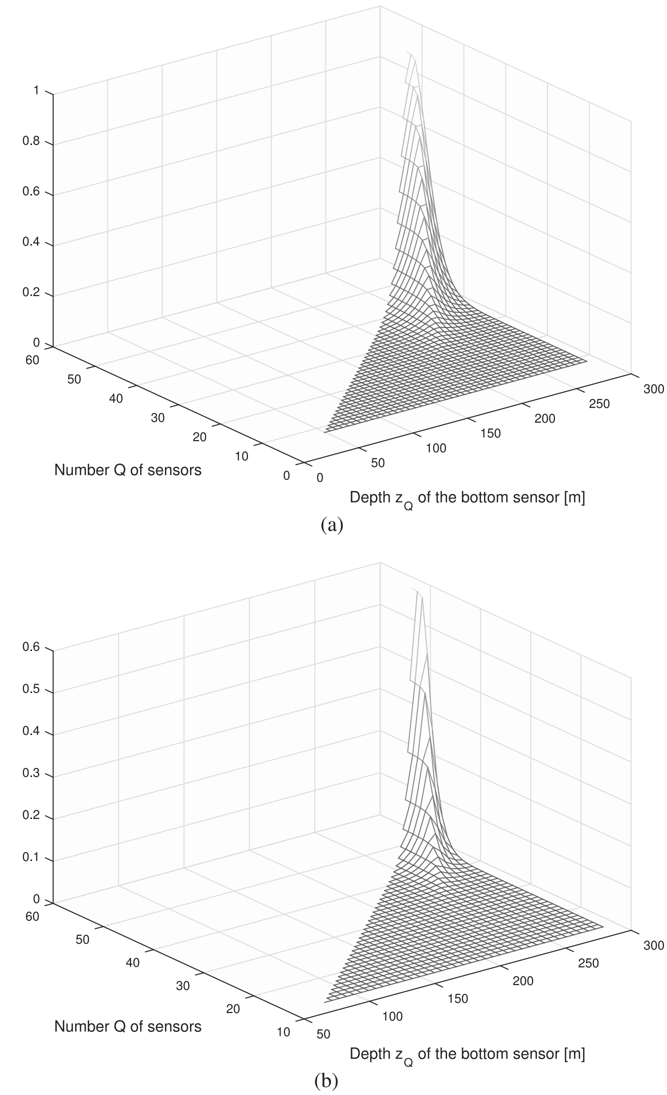


Fig. 2. Inverse of the condition number of the matrix $\Phi(z_1,\ldots,z_Q)$ of modal functions corresponding to a dense VLA with sensors spaced by 5.6 [m], as function of the number Q of sensors, and the depth z_Q of the bottom sensor. (a) 8-mode waveguide. (b) 13-mode waveguide.

C. Noise-Corrupted Data

As described above, the algorithm delivers exact values of the normal mode parameters, if there is no noise. Noise affects performance, making the algorithm estimates deviate from exact values. The presence of noise calls for some adaptations to the procedure introduced earlier. While, in the absence of noise, we can interchangeably use $(k,l)=(1,2)$ or $(k,l)=(2,1)$; in practice, a VLA that is closer to the source collects data with larger SNR. Hence, if $R_1 < R_2$, then $\mathbf{Y}_1$ is presumed to be more accurately measured than $\mathbf{Y}_2$. It is, then, the one we decide to invert, to eventually calculate $\mathbf{Y}_2\mathbf{Y}_1^\natural$, which is equal to $\sqrt{R_1/R_2}\Phi\mathrm{Diag}\left(e^{j\kappa_1(R_1-R_2)},\ldots,e^{j\kappa_M(R_1-R_2)}\right)\Phi^\natural$, in the noise-free case.

Also, as mentioned at the end of Section II, $\mathbf{P}_k \hat{=} \mathbf{P}(R_k)$, $k=1,2$ are symmetric. Therefore, if we update $\mathbf{Y}_k$ as $(\mathbf{Y}_k + \mathbf{Y}_k^T)/2$, then $\mathbf{Y}_k = \mathbf{P}_k + [\mathbf{E}(R_k) + \mathbf{E}^T(R_k)]/2$, for $k=1,2$, in which the transformed noise $[\mathbf{E}(R_k)+\mathbf{E}^T(R_k)]/2$ has, under some conditions not detailed here, half the power of the original noise component $\mathbf{E}(R_k)$.

Assuming measurements are collected in the presence of noise, the estimation algorithm is executed as follows.

1) Collect $Q \times Q$ matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$.
2) Update $\mathbf{Y}_k$ as $(\mathbf{Y}_k + \mathbf{Y}_k^T)/2$ for $k=1,2$.
3) For $m=1,\ldots,M$, let σ_m be the largest singular value of $\mathbf{Y}_1$. Let $\mathbf{u}_m$ and $\mathbf{v}_m$ be, respectively, the associated left and right unit-norm singular vectors.
4) Calculate $\mathbf{Y}_1^\natural$ as $\sum_{m=1}^{M}\sigma_m^{-1}\mathbf{v}_m\mathbf{u}_m^H$.
5) For $m=1,\ldots,M$, let λ_m be the eigenvalue of $\mathbf{Y}_2\mathbf{Y}_1^\natural$, with the mth largest magnitude, and associated to the unit-norm eigenvector $\mathbf{w}_m$.
6) Estimate $(R_1-R_2)\kappa_m$ as $\arg(\lambda_m)$, selected in $[0,2\pi]$.
7) Estimate $[\phi_m(z_1),\ldots,\phi_m(z_Q)]^T$ by $\mathbf{w}_m$.

At this point, it is worth noting that the obtained estimates of the modal functions and the wave numbers do suffer from some indeterminacy. Such scale and phase ambiguities are common in subspace techniques [7], but also in transform-domain techniques [2]. While it is possible to fix such ambiguities [6, Sec. II.C], [7], we prefer not to address this issue in this paper, and, instead, to test and evaluate our technique in a way that is

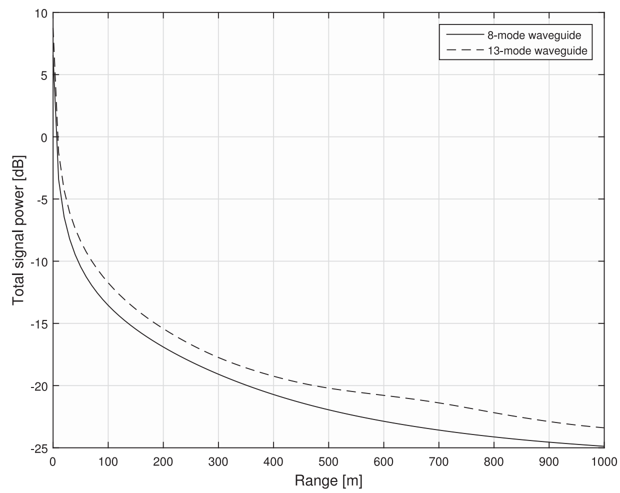


Fig. 3. Total signal power collected by a VLA as function of its range to the source. The 37 sensors are deployed at depths $60.4, 66, \ldots, 262$ [m].

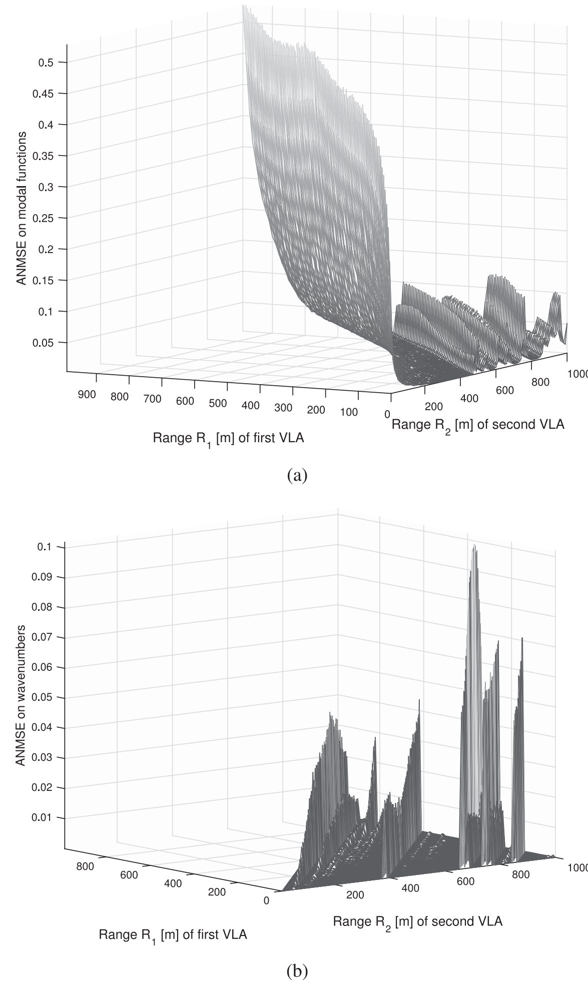


Fig. 4. Estimation accuracy as function of the range of the VLAs made, each, of 37 sensors deployed at depths $60.4, 66, \ldots, 262$ [m]. The 8-mode waveguide is corrupted by an AWGN characterized by a standard deviation $\sigma_n = 10^{-4}$.

independent from the ambiguity removal technique. This choice is also motivated by the fact that many applications can accommodate ambiguities to a certain extent (as in communication systems, for example), to the point where only sign estimation is critical [2, Sec. III.B.2].

D. Identifiability

As a subspace technique, the proposed algorithm is based on rank properties of matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2\mathbf{Y}_1^\natural$. In particular, their respective SVD and EVD are critical to both estimation accuracy and computation complexity. For instance, measurement noise will be amplified if either matrices is near singular. This would happen if different modes exhibit similar wave numbers and/or similar modal functions.

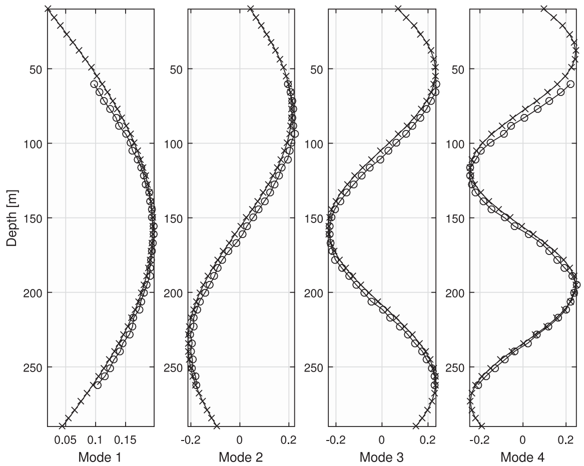
In the following, we list the conditions required for exact estimation of the modal parameters. Each time, we discuss first, how the nonfulfillment of the condition impacts estimation accuracy, and second, what counter-measures can be adopted. Exact parameter estimation is obtained if all of the following conditions are verified.

1) $\mathbf{Y}_k$ has rank M. This, in turn, requires the following.
   a) Φ is tall, or $Q \geq M$. Near-singularity may occur, however, as in the case of reflection-less bottom [15, Sec. VI]. This can be attenuated by changing the antenna size, aperture, and depth, since we are sure that when the VLA tends to cover the entire water column, then columns of Φ tend to be orthogonal.
   b) $\mathbf{A}(r)$ is invertible. However, $\mathbf{A}(r)$ can be near-singular if $\max \kappa_m \gg \min \kappa_m$, i.e., coexistence of very weak and very strong modes.
2) Eigenvalues of $\mathbf{Y}_2\mathbf{Y}_1^\natural$ have, each, multiplicity one: Larger-than-one multiplicity occurs if two distinct wave numbers happen to verify $\kappa_m(R_1 - R_2) = \kappa_n(R_1 - R_2)$. As a counter-measure, we can consider to move the VLAs to different locations where, expectedly, $(\kappa_m - \kappa_n)(R_1 - R_2)$ is not a multiple of 2π for any mode.
3) Noise-free measurements: The effect of noise can be reduced by moving the VLA closer to the source and/or time-averaging the measurements collected while the source is maintained at a fixed depth. Matrix perturbation theory [24] allows us to claim that estimates obtained at low-level noise will be close to those obtained at no noise, i.e., close to exact parameters. Hence, we expect the proposed technique to be asymptotically (for an increasing SNR) unbiased. We will comfort this intuition with experimental results.

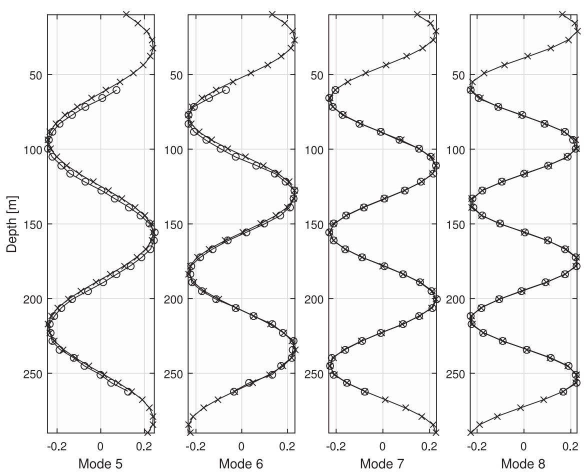
IV. Simulations

A. Simulated Environments

We test two 300 [m] deep waveguides excited by a monochromatic source emitting a tone at 60 [Hz]: first, a Pekeris waveguide supporting $M = 8$ acoustic modes, and second, a lossy waveguide supporting $M = 13$ acoustic modes with 10 [m] thick sediment layer. We refer to the two environments as the 8-mode waveguide and the 13-mode waveguide, respectively.

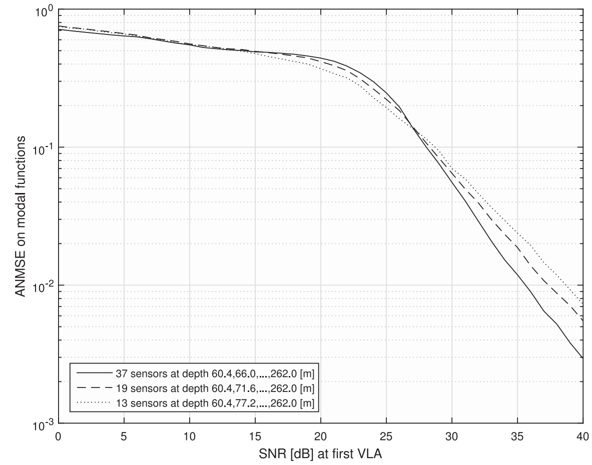


(a)

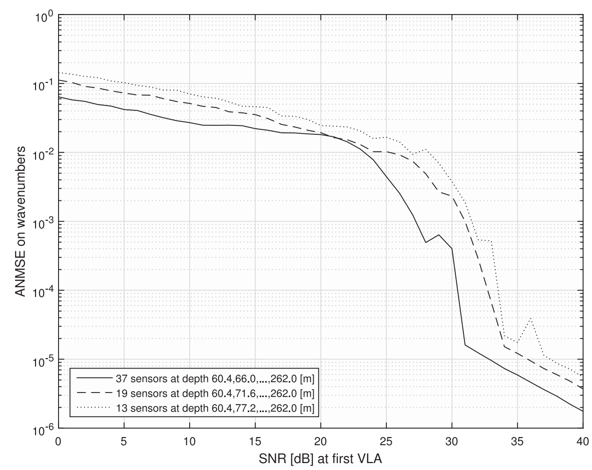


(b)

Fig. 5. Exact (“x”-dots) and estimated (“o”-dots, after amplitude and sign corrections) modal functions of the 8-mode waveguide. The two VLAs are placed at 200 [m] and 400 [m] from the source, with respective SNRs of 31.7384 and 27.9169 [dB], corresponding to $\sigma_n = 10^{-4}$.



(a)



(b)

Fig. 6. Estimation accuracy for the 8-mode waveguide, as function of the observation SNR and the intersensor spacing, for two VLAs placed at 200 and 400 [m] from the source. The SNR at the second VLA is 3.8215 (resp., 3.7829 and 3.7435) [dB] lower than that at the first VLA, when sensors are spaced by 5.6 (resp., 11.2 and 16.8) [m].

Dimensions and acoustic properties are shown in Fig. 1(a), while transmission loss is shown in Fig. 1(b).

Using the normal-mode modeling software KRAKEN, we generate snapshots $p(z_q, z_{q'}, r)$, for $z_q, z_{q'} = 10, 15.6, \ldots, 290$ [m] and $r = 10, 20, \ldots, 1000$ [m]. For a VLA placed at range r with its sensors at depths $z_1, \ldots, z_Q$, the power of the collected signal is calculated as follows. Given that the source is activated at, consecutively, depths $z_1, \ldots, z_Q$, we have a total of Q^2 measurements $p(z_q, z_{q'}, r)$, for $z_q, z_{q'} = z_1, \ldots, z_Q$, grouped into $Q \times Q$ matrix $\mathbf{P}(r)$. The average (per sensor) signal power is given by $(1/Q^2) \sum_{q,q'=1}^{Q} |p(z_q, z_{q'}, r)|^2$, which depends on the actual positions of the sensors.

The fact that the 13-mode waveguide is a more severe environment is illustrated in Fig. 2, where we report the condition number, defined as the ratio of the largest to the smallest singular value [23], of matrix $\Phi(z_1, \ldots, z_Q)$. The lower the condition number, the easier is the identification problem, with most (subspace) techniques performing well for a condition number lower than 10. As we vary the array size Q and depth z_Q (of the bottom sensor), we realize that the matrix $\Phi(z_1, \ldots, z_Q)$ is better conditioned more frequently for the 8-mode waveguide than for the 13-mode waveguide. Equally important is the power collected by the VLA, which, as illustrated in Fig. 3, shows to deteriorate quickly as the VLA moves away from the source. Luckily, the proposed technique is based on a property that is valid throughout the waveguide, and so, it can benefit from larger SNRs available at shorter ranges. In contrast, the work in [2] and [9] is based on a (mode separation) property that is encountered rather far from the source, where large SNR is hardly encountered.

B. Simulation Results

To study the performance of the proposed estimation algorithm in a noisy environment, we corrupt KRAKEN generated pressure field with zero-mean complex-valued circular additive white Gaussian noise (AWGN), following (5). We

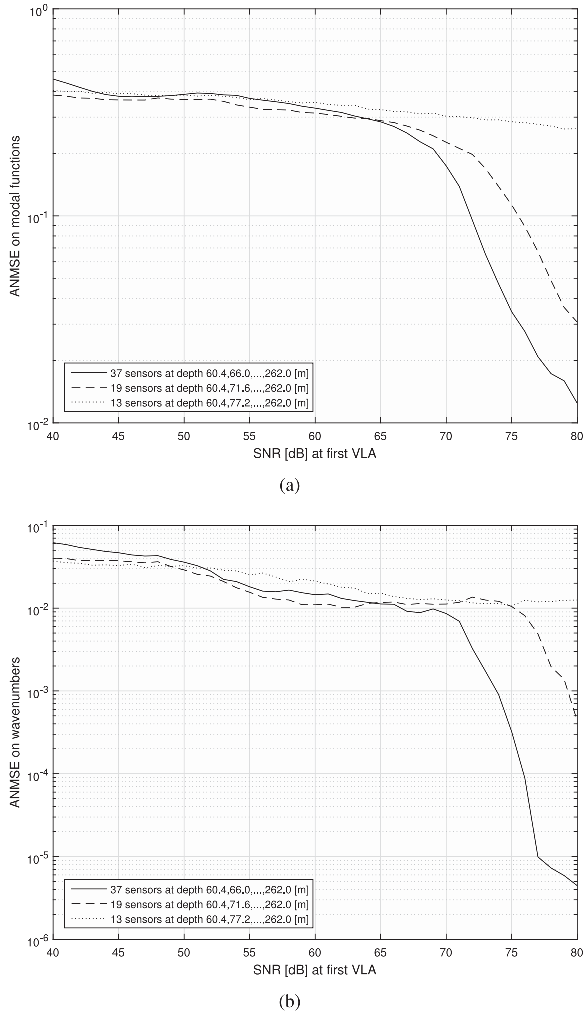


Fig. 7. Estimation accuracy for the 13-mode waveguide, as function of the observation SNR and the intersensor spacing, for two VLAs placed at 200 and 400 [m] from the source. The SNR at the second VLA is 3.8217 (resp., 3.7467 and 3.6655) [dB] lower than that at the first VLA, when sensors are spaced by 5.6 (resp., 11.2 and 16.8) [m].

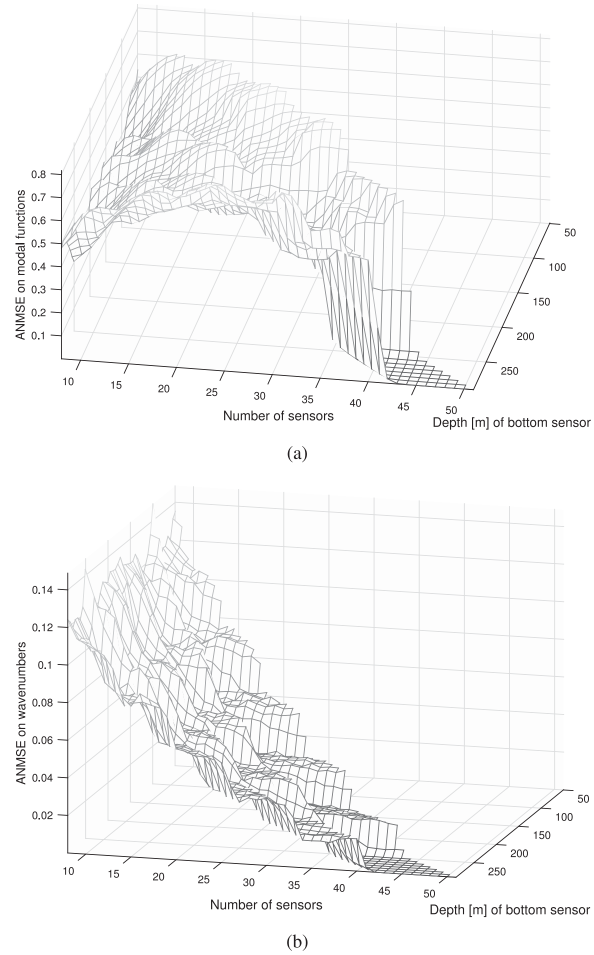


Fig. 8. Estimation accuracy for dense VLAs of all possible sizes and positions, deployed at 200 [m] and 400 [m] from the source. The 8-mode waveguide is corrupted by an AWGN characterized by a standard deviation $\sigma_n = 10^{-4}$.

assume noise components to be independent at every sensor, but with equal power $\mathrm{E}\left[|\epsilon(z^S, z_q, r)|^2\right]$, regardless of the source/receiver positions [15]. Subsequently, for a VLA deployed at range r, the reported SNR is defined as the one obtained by averaging over all sensors, and that we calculate as $[1/(Q^2\sigma_n^2)]\sum_{q,q'=1}^{Q}|p(z_q, z_{q'}, r)|^2$ [15], [16, Sec. VI].

The presence of noise will result in an estimate $\mathbf{w}_m$ that is roughly colinear to the exact mth modal function $[\phi_m(z_1), \ldots, \phi_m(z_Q)]^T$. It is custom, when a vector $\mathbf{x}$ is estimated by $\hat{\mathbf{x}}$ up to an unknown multiplicative constant, to use the following normalized mean square error (MSE) defined as $\left(1/\|\mathbf{x}\|^2\right)\min_\beta \|\mathbf{x} - \beta\hat{\mathbf{x}}\|^2$, and proved in [25] to be equal to $1 - \left[\left|\mathbf{x}^H\hat{\mathbf{x}}\right|/(\|\mathbf{x}\|\|\hat{\mathbf{x}}\|)\right]^2$. This is a normalized error in the sense that it ranges from 0 ($\mathbf{x}$ and $\hat{\mathbf{x}}$ are co-linear) to 1 ($\mathbf{x}$ and $\hat{\mathbf{x}}$ are orthogonal). Applied to our estimates of the M modal functions, a global normalized performance measure is calculated as

$$1 - \frac{1}{M}\sum_{m=1}^{M}\left(\frac{|[\phi_m(z_1), \ldots, \phi_m(z_Q)]\,\mathbf{w}_m|}{\|\,[\phi_m(z_1), \ldots, \phi_m(z_Q)]\,\|\|\mathbf{w}_m\|}\right)^2.$$

This normalized MSE on modal functions is averaged over 1000 Monte Carlo runs, and reported in the figures as the ANMSE, for the averaged normalized MSE.

As described earlier, the algorithm delivers $\arg(\lambda_m)$ as an estimate of $(R_1 - R_2)\kappa_m$, modulo 2π, which does not uniquely characterize the mth wave number. To assess performance in a way that does not depend on how we solve this ambiguity problem, both $\arg(\lambda_m)$ and $(R_1 - R_2)\kappa_m, m = 1, \ldots, M$, are evaluated modulo 2π. As a performance indicator, reported in the different figures, we calculate $(1/M)\sum_{m=1}^{M}[(R_1 - R_2)\kappa_m - \arg(\lambda_m)]^2$ that we, abusively, term as ANMSE on wave numbers.

The two performance measures are reported in Fig. 4 as function of the location of the two VLAs. As one would expect, the closer the VLAs to the source, the better is the array performance. We notice that errors also show to depend mostly on $R_1 - R_2$, as predicted by the expression (6) of the key matrix $\mathbf{Y}_2\mathbf{Y}_1^\natural$.

Inspired by these results, we decided to place the two VLAs at 200 and 400 [m] from the source. We show sample estimates of the 8 modes in Fig. 5, confirming the ability of the algorithm to deliver accurate estimates.

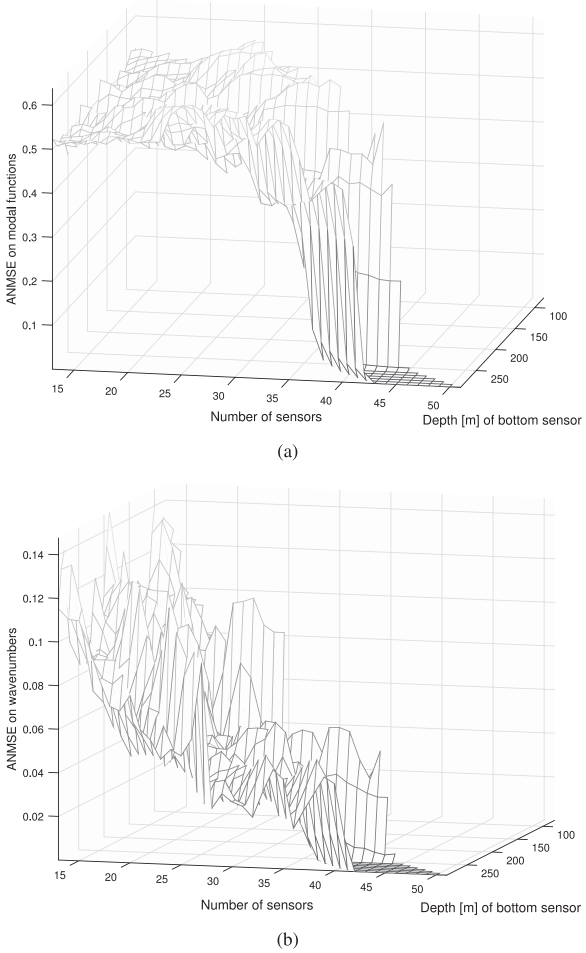


Fig. 9. Estimation accuracy for dense VLAs of all possible sizes and positions, deployed at 200 [m] and 400 [m] from the source. The 13-mode waveguide is corrupted by an AWGN characterized by a standard deviation $\sigma_n = 10^{-6}$.

This is further confirmed by Fig. 6, where we test different array sizes and intersensor spacings, but also different noise levels. Low ANMSE (below 10%) is obtained for SNR of 25 [dB] or more, for the 8-mode waveguide. This is comparable to the SNR required by other algorithms [1], [8], [9], [11] to deliver good estimates. However, the work in [9] requires such SNR to be available at 1 [km] or farther, which is unlikely to be the case. The more selective 13-mode waveguide turns to be much more demanding, in terms of SNR and array size, as shown in Fig. 7.

Finally, with the two VLAs always placed at ranges 200 and 400 [m], we vary their size and depth in Figs. 8 and 9, for the 8-mode waveguide and the 13-mode waveguide, respectively. Again, low ANMSE is obtained for longer arrays (of 35 sensors or more) deployed deeper in the water column.

V. Conclusion

Subspace techniques are more attractive than search-based transformation-domain techniques, especially when they deliver closed-form estimates. At the same time, they may suffer from possible matrix degeneracy and, in the case of normal mode estimation, from the requirement that the array should fully span the water column. In this paper, we develop a subspace algorithm that manipulates a nontrivially derived data matrix, instead of the brute array output data matrix. This original data matrix combines measurements taken by two nontowed VLAs covering partially and sparsely the water column. This matrix has an interesting eigenstructure where modal functions and wave numbers appear, respectively, as eigenvectors and eigenvalues. EVD of this matrix delivers closed-form estimates (of modal functions and wave numbers) that tend to be exact when observation noise tends to zero.

Appendix

Let us consider $\mathbf{Y}_k\mathbf{Y}_l^\natural$ as expressed in (6), where the $Q \times M$ matrix Φ is full column rank. For $m = 1, \ldots, M$, we designate σ_m to be the nonzero mth singular value of Φ. By SVD, we obtain unitary matrices $\mathbf{U}$ and $\mathbf{V}$ such that

$$\Phi = \mathbf{U}\begin{bmatrix}\mathrm{Diag}(\sigma_1, \ldots, \sigma_M)\\ \mathbf{0}\end{bmatrix}\mathbf{V}^H$$

and form the pseudoinverse

$$\Phi^\natural = \mathbf{V}\left[\mathrm{Diag}(\sigma_1^{-1}, \ldots, \sigma_M^{-1})\mathbf{0}\right]\mathbf{U}^H.$$

If we split $\mathbf{U} = [\mathbf{U}_1\mathbf{U}_2]$, with $\mathbf{U}_2$ gathering all $M - Q$ unit-norm singular vectors left orthogonal to Φ, then we also have

$$\Phi = \mathbf{U}_1\mathrm{Diag}(\sigma_1, \ldots, \sigma_M)\mathbf{V}^H$$

and

$$\Phi^\natural = \mathbf{V}\mathrm{Diag}(\sigma_1^{-1}, \ldots, \sigma_M^{-1})\mathbf{U}_1^H.$$

To make it square, we complete Φ as $\Phi^+ \hat{=} [\Phi\mathbf{U}_2]$. Of course, $\mathbf{U}_2$ is left orthogonal to $\mathbf{U}_1$, and so to both Φ and $(\Phi^\natural)^H$. Consequently, Φ^+ is invertible by $[\Phi^+]^{-1} \hat{=} \left[\begin{smallmatrix}\Phi^\natural\\ \mathbf{U}_2^H\end{smallmatrix}\right]$, in which, case (6) is updated as follows:

$$\mathbf{Y}_k\mathbf{Y}_l^\natural = \Phi^+\begin{bmatrix}\mathbf{A}(\mathbf{R_k})\mathbf{A^{-1}}(\mathbf{R_l}) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\left[\Phi^+\right]^{-1}$$

or, to further highlight the eigenstructure

$$\begin{aligned}&\mathbf{Y}_k\mathbf{Y}_l^\natural\Phi^+\\ &= \sqrt{\frac{R_l}{R_k}}\Phi^+\begin{bmatrix}\mathrm{Diag}\left(e^{j\kappa_1(R_l-R_k)}, \ldots, e^{j\kappa_M(R_l-R_k)}\right) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}.\end{aligned}$$

From the above, columns $1, \ldots, Q$ of Φ^+ are eigenvectors of $\mathbf{Y}_k\mathbf{Y}_l^\natural$, associated to eigenvalues

$$\sqrt{\frac{R_l}{R_k}}e^{j\kappa_1(R_l-R_k)}, \ldots, \sqrt{\frac{R_l}{R_k}}e^{j\kappa_M(R_l-R_k)}, 0, 0, \ldots,$$

respectively. In particular, if $e^{j\kappa_1(R_l-R_k)}, \ldots, e^{j\kappa_M(R_l-R_k)}$ are distinct, then eigenvalues have each multiplicity one, and the respective eigenvectors are equal (up to a multiplicative constant) to columns $1, \ldots, M$ of Φ.

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