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Range-Dependent Regularization of Travel Time Tomography based on Theoretical Modes

Orlando C. Rodríguez, Sérgio M. Jesus
Signal Processing Laboratory (SiPLAB), FCT - Universidade do Algarve, Campus de Gambelas, 8000 – Faro, Portugal. orodrig@ualg.pt, sjesus@ualg.pt

Summary
Travel time inversion is a fundamental method of Ocean Acoustic Tomography, to estimate perturbations in sound speed. By discretizing the watercolumn into a system of layers, the method allows to introduce a system of linear equations, relating a known vector of perturbations in travel time, to an unknown vector of perturbations in sound speed, through the so-called “observation matrix”. Inverting the system allows to estimate the perturbation in sound speed in each layer of the watercolumn. However, in most problems of practical interest, the number of unknowns (i.e. the perturbations in sound speed) is larger that the number of equations (i.e. the number of delays in travel time). Thus, inverting the system can be viewed as an ill-posed problem. The discussion presented in this paper illustrates an approach to the inversion problem, which is based on the usage of theoretical modes. Further, it is shown that for a range-dependent perturbation in sound speed, corresponding to a superposition of plane waves, the inversion problem can be regularized (i.e. the system can be rewritten in order to deal with more equations than unknowns) by estimating only the amplitudes and phases of the linear waves. Particular examples are given for real data.

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1. Introduction

Travel time inversion is a fundamental method of Ocean Acoustic Tomography, that allows to introduce a system of linear equations, relating a known vector of perturbations in travel time, to an unknown vector of perturbations in sound speed, through the so-called “observation matrix”. Inverting the system allows to determine the pertubations in sound speed, by estimating it in each layer of the watercolumn. In most cases inverting the system can be viewed as an ill-posed problem since the number of unknowns uses to be larger than the number of equations. Theoretical modes (hereafter TMs) can be used to regularize the problem of inversion, by allowing to rewrite the system in order to obtain more equations than unknowns (see a preliminary discussion with application to real data in [1]). This paper explores the regularization based on TMs by developing a range-dependent inversion of sound speed, for the case of internal plane-wave propagation. Thus, by inverting the system one estimates the amplitudes and phases of plane waves. The feasibility of the method is tested on real data acquired during the INTIMATE’96 experiment.

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2. Theoretical Background

2.1. Travel time inversion

Briefly, as discussed in [2], for a small change of sound speed, $\delta c(z) = c(z) - c_0(z) \ll c_0(z)$, the perturbation in travel time, $\Delta\tau_j$, of an acoustic pulse can be written as

$$\Delta\tau_j = \tau_j - \tau_j^0 = \int_{\Gamma_j} \frac{\mathrm{d}s}{c(z)} - \int_{\Gamma_j} \frac{\mathrm{d}s}{c_0(z)} \approx -\int_{\Gamma_j} \frac{\delta c(z)}{c_0^2(z)} \mathrm{d}s, \tag{1}$$

where the integral in equation (1) is taken along the unperturbed eigenray Γ_j. For a set of T perturbations in travel time, and discretizing the watercolumn into a system with L layers, one can relate a vector of delays, $\boldsymbol{\Delta\tau}$, to a vector of perturbations in sound speed, $\boldsymbol{\delta}\mathbf{c}$, through a linear system of equations:

$$\boldsymbol{\Delta\tau} = \mathbf{E}\boldsymbol{\delta}\mathbf{c} + \mathbf{n}, \tag{2}$$

where $\boldsymbol{\Delta\tau} = [\Delta\tau_1, \Delta\tau_2, \ldots, \Delta\tau_\mathsf{T}]^\mathsf{t}$,
$\boldsymbol{\delta}\mathbf{c} = [\delta c_1, \delta c_2, \ldots, \delta c_\mathsf{L}]^\mathsf{t}$,

where each δc_l corresponds to an average of $\delta c(z)$, in the lth layer; $[\ldots]^\mathsf{t}$ represents the transpose of vector $[\ldots]$. In equation (2) $\mathbf{n}$ accounts for rounding errors, and for statistic contributions from noise sources. Matrix $\mathbf{E}$, dimension $\mathsf{T} \times \mathsf{L}$, is known as the *Observation Matrix* and can be calculated from unperturbed eigenrays. In most cases of practical interest $\mathsf{L} \gg \mathsf{T}$, so equation (2) corresponds to an undetermined system. Providing that rank$\mathbf{E} = \mathsf{T}$ one can select the minimum norm solution [3]:

$$\boldsymbol{\delta}\mathbf{c}^{\#} = \left(\mathbf{E}\mathbf{E}^{\mathsf{t}}\right)^{-1} \mathbf{E}^{\mathsf{t}} \boldsymbol{\Delta\tau}. \tag{3}$$

2.2. Multiple hydrophones

The treatment of a system of equations for N hydrophones $\boldsymbol{\Delta\tau}_1 = \mathbf{E}_1\boldsymbol{\delta}\mathbf{c} + \mathbf{n}_1$, $\boldsymbol{\Delta\tau}_2 = \mathbf{E}_2\boldsymbol{\delta}\mathbf{c} + \mathbf{n}_2$, …, $\boldsymbol{\Delta\tau}_{\mathsf{N}} = \mathbf{E}_{\mathsf{N}}\boldsymbol{\delta}\mathbf{c} + \mathbf{n}_{\mathsf{N}}$, sharing a common vector $\boldsymbol{\delta}\mathbf{c}$, can be handled by introducing the following concatenated vectors and matrices:

$$\boldsymbol{\Delta\tau} = \begin{bmatrix} \boldsymbol{\Delta\tau}_1 \\ \boldsymbol{\Delta\tau}_2 \\ \vdots \\ \boldsymbol{\Delta\tau}_{\mathsf{N}} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_{\mathsf{N}} \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \vdots \\ \mathbf{n}_{\mathsf{N}} \end{bmatrix}, \tag{4}$$

and further applying the solution equation (3).

2.3. Theoretical modes

For the hydrostatic linear rotationless case the set of TMs, Ψ_m, can be calculated by solving a Sturm-Liouville problem [1]:

$$\frac{\mathrm{d}^2\Psi_m}{\mathrm{d}z^2} + \frac{N^2}{C_m^2}\Psi_m = 0 + \mathrm{BC}, \tag{5}$$

where BC means “Boundary Conditions”. $N^2(z)$ represents the buoyancy frequency, which can be calculated using temperature [4]. In equation (5) C_m represents the propagation velocity of the mth linear wave; for a fixed frequency ω, the wavenumber will correspond to $k_m = \omega/C_m$. Under homogeneous top and bottom BCs the TMs form an orthogonal basis of functions: $\langle \Psi_m \left| N^2 \right| \Psi_n \rangle = 0$ for $m \neq n$, where $\langle f_1 \left| f_2 \right| f_3 \rangle = \int_0^D f_1 f_2 f_3 \, \mathrm{d}z$.

2.4. Plave-wave propagation

For internal plane-wave propagation the perturbation on sound speed corresponds to

$$\delta c\,(z,r) = \frac{\mathrm{d}c_0}{\mathrm{d}z} \sum_{m=1}^{M} \Psi_m(z) \big[\alpha_m \sin(k_m r \cos\theta) + \beta_m \cos(k_m r \cos\theta), \tag{6}$$

where θ represents the direction of propagation (see Figure 1a). M represents the number of relevant TMs.

2.5. Regularization using theoretical modes

Substituting equation (6) into equation (1) one obtains the system

$$\boldsymbol{\Delta\tau} = \mathbf{P}\mathbf{x} + \mathbf{n}, \tag{7}$$

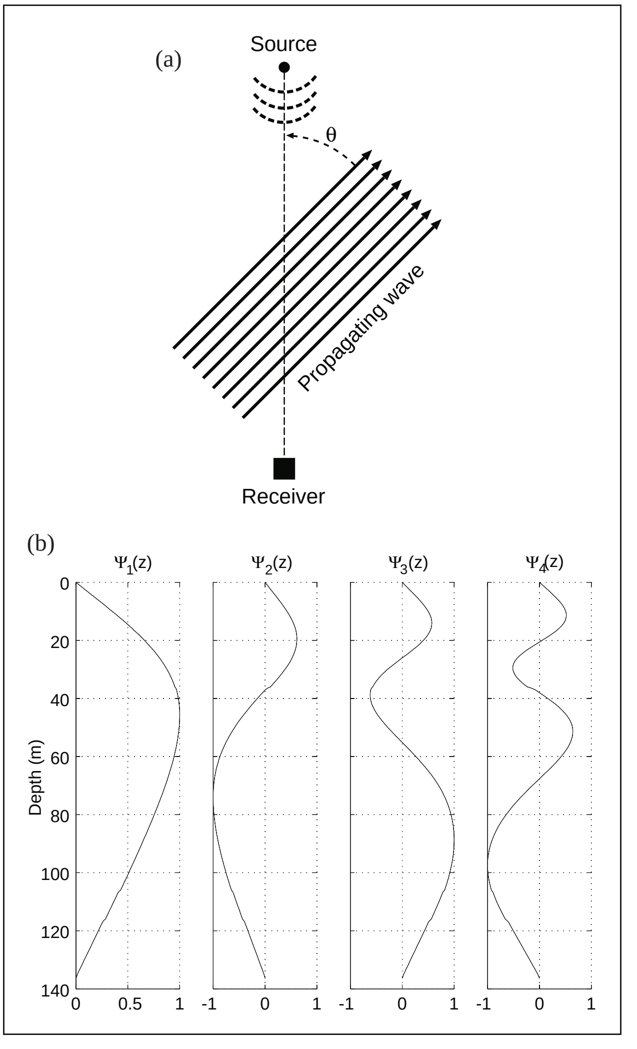

Figure 1. (a) Propagating direction of the internal plane wave related to the acoustic path; (b) temperature-derived theoretical modes $\Psi_m(z)$ (INTIMATE’96 hydrographic data).

where $\mathbf{P} = [\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{R}}]$, and $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{R}}$ correspond to

$$\boldsymbol{\mathcal{S}} = \begin{bmatrix} \boldsymbol{\mathcal{S}}_1^{\mathsf{t}} \\ \boldsymbol{\mathcal{S}}_2^{\mathsf{t}} \\ \vdots \\ \boldsymbol{\mathcal{S}}_{\mathsf{T}}^{\mathsf{t}} \end{bmatrix}, \qquad \boldsymbol{\mathcal{R}} = \begin{bmatrix} \boldsymbol{\mathcal{R}}_1^{\mathsf{t}} \\ \boldsymbol{\mathcal{R}}_2^{\mathsf{t}} \\ \vdots \\ \boldsymbol{\mathcal{R}}_{\mathsf{T}}^{\mathsf{t}} \end{bmatrix},$$
$$\boldsymbol{\mathcal{S}}_j^{\mathsf{t}} = [\mathcal{S}_{j1}, \mathcal{S}_{j2}, \ldots, \mathcal{S}_{jM}], \tag{8}$$
$$\boldsymbol{\mathcal{R}}_j^{\mathsf{t}} = [\mathcal{R}_{j1}, \mathcal{R}_{j2}, \ldots, \mathcal{R}_{jM}].$$

Further, the vector of modal amplitudes $\mathbf{x}$ corresponds to

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} \text{ where } \boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{bmatrix}, \; \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_M \end{bmatrix}, \tag{9}$$

and

$$\begin{bmatrix} \mathcal{S}_{jm} \\ \mathcal{R}_{jm} \end{bmatrix} = \int_{\Gamma_j} \frac{\mathrm{d}c_0}{\mathrm{d}z} \frac{\Psi_m(z)}{c_0^2} \mathrm{d}s \times \begin{bmatrix} \sin(k_m r \cos\theta) \\ \cos(k_m r \cos\theta) \end{bmatrix}. \tag{10}$$

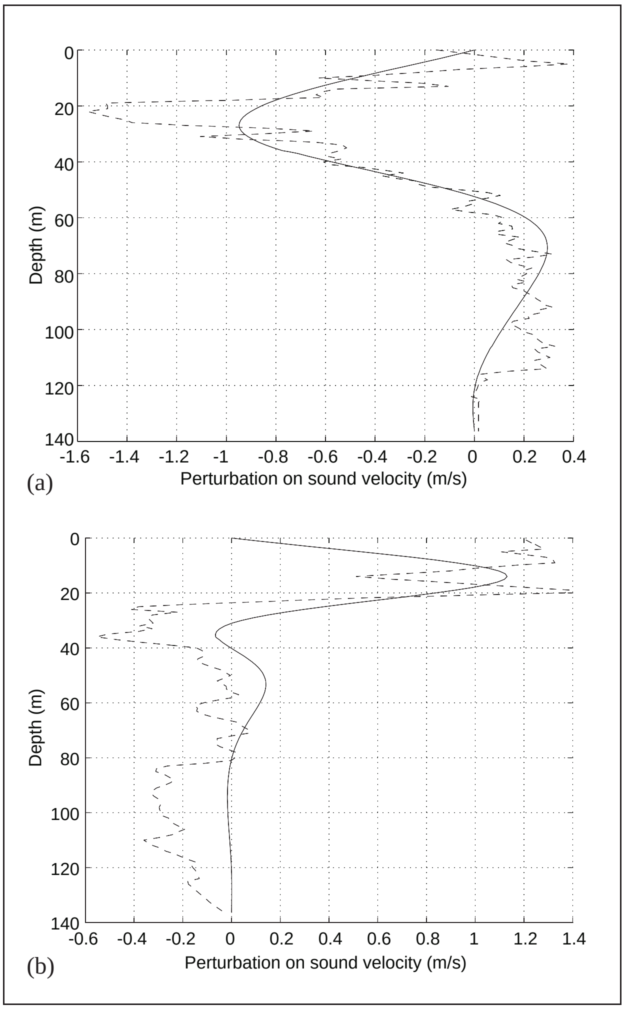

Figure 2. (a) Best match of sound speed perturbations near the VLA; (b) extrapolated profile of sound speed perturbations near the position of the acoustic source. In both cases the continuous line corresponds to the estimated value, while the dashed line represents direct measurements.

Providing that $2M < \mathsf{T}$ one can regularize equation (2) and write the solution of the system equation (7) as [3]

$$\mathbf{x}^{\#} = \left(\mathbf{P}^{\mathrm{t}}\mathbf{P}\right)^{-1}\mathbf{P}^{\mathrm{t}}\boldsymbol{\Delta\tau}. \tag{11}$$

The solution equation (11) allows to determine $\mathbf{x}$, which determine uniquely $c(z,r) = c_0(z) + \delta c(z,r)$, through equation (6). A multiple hydrophone system can be handled efficiently by concatenating once the corresponding systems of equations.

3. Application to Real Data

The range-dependent regularization was tested on acoustic data from the INTIMATE'96 experiment [1]. TMs were calculated from acquired CTD data (see Figure 1b). The propagation geometry corresponded to a source at 90 m, a Vertical Line Array (hereafter VLA) with three hydrophones (at 35, 105 and 115 m), a range R = 5,6 km and a bottom depth D = 135 m. Stable eigenrays were calculated using an average profile $c_0(z)$. It was considered θ = 75° [4]. Due to synchronization problems absolute arrivals were not available. To compensate it was developed an accurate estimation of bottom depth, and an accurate match minimizing $\left|\left|\boldsymbol{\delta}\mathbf{c}^{\#} - \boldsymbol{\delta}\mathbf{c}\right|\right|$, where $\boldsymbol{\delta}\mathbf{c}^{\#}$ was a *range-independent* estimate of $\boldsymbol{\delta}\mathbf{c}$. That matching was repeated once more, based on equation (6), with different range discretizations. Arrival redundancy ([5]) imposed the constraint that M = 4. The optimized match (which allowed to estimate $\boldsymbol{\beta}$) can be seen on Figure 2a, and shows an accurate agreement between true and estimated profiles. Additional tests of optimization for slight variations of θ around 75° provided unrealistic estimates of $\mathbf{x}$. The estimate of $\boldsymbol{\alpha}$ was found by optimizing

$$\boldsymbol{\alpha}^{\#} = \left(\boldsymbol{\mathcal{S}}^{\mathrm{t}}\boldsymbol{\mathcal{S}}\right)^{-1}\boldsymbol{\mathcal{S}}^{\mathrm{t}}\left(\boldsymbol{\Delta\tau} - \boldsymbol{\mathcal{R}}\beta^{\#}\right), \tag{12}$$

for different range discretizations, and choosing the estimate with a minimum norm. The corresponding profile $\delta c(z,R)$ can be seen on Figure 2b, and reveals a good agreement with direct measurements of $\delta c(z)$, taken near the position of the acoustic source.

4. Conclusions

The feasibility of range dependent regularization based on TMs was tested on real data for the case of internal plane-wave propagation in a shallow water environment. Inversion results proved to be accurate, the inversion procedure was found to be robust and able to resolve high order TMs. However, the numerical evaluation of $\mathbf{P}$ still remains an open question, and the inversion is limited to the internal plane-waves, crossing the experimental scenario at a fixed direction.

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